Proof of the Extended Delta Conjecture

Review of Sections 2 and 5

Extended Delta Conj: Let $\ell, k \in \mathbb{N}$, $k \leq n-1$

Then $\Delta_{h_\ell} \Delta'_{e_{n-1-k}} e_n$

ignore 0's

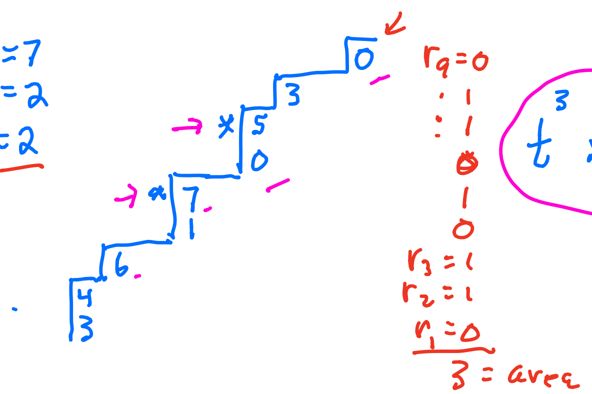
$$= \langle z^k \rangle \sum_{\substack{\pi \in D_{n+\ell} \\ P \in W\widetilde{PF}(\pi)}} q^{dinv(P)} t^{area(P)} x^{wt_+(P)} \prod_{r_i = r_{i-1}+1} \left(1 + \frac{z}{t^{r_i}}\right)$$

coef of z^k in

(ℓ 0's in P)
(no 0 in bottom row)

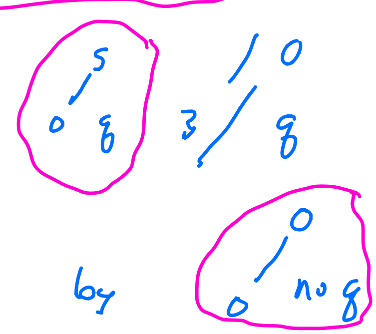
r_i = length of ith row (from the bottom)

ex. $n=7$
$\ell=2$
$k=2$



dinv
3 : 1 6 : 4
5 : 2 4 : 3
7 : 4

$t^3 x_1 x_3^2 x_4 x_5 x_6 x_7 \, q^{14}$



dinv : same for positive cars
obvious extension to 0-cars by
standardizing

Remark: Adding ℓ 0's to a WPF, is essentially the same as taking $h_\ell^\perp$ of $\nabla e_{n+\ell}$

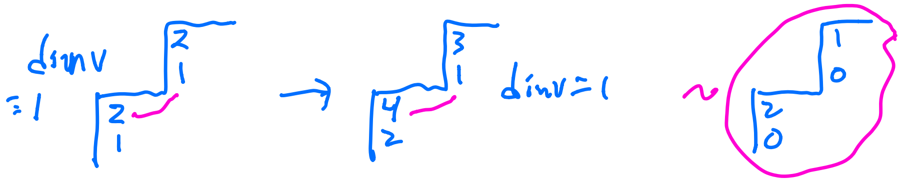
where $\langle f, gh\rangle = \langle g^{\perp} f, h\rangle$ all $f,g,h \in \Lambda$

To see why, note

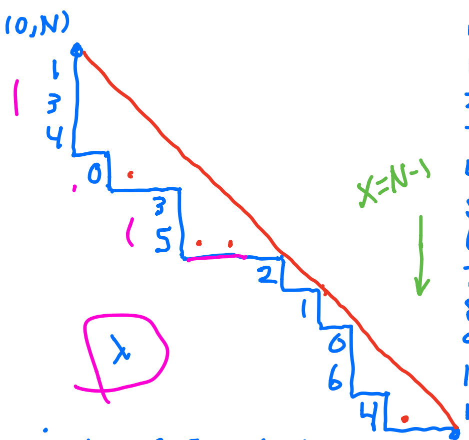
$$\langle \nabla e_{n+\ell}, h_\ell h_\lambda \rangle \qquad \lambda \vdash n$$

$$= \nabla e_{n+\ell}\Big|_{m_{\ell,\lambda}} = \sum_{\substack{P \in WPF \\ \ell\ 1\text{'s}\ \lambda_1\ 2\text{'s}\ \lambda_2\ 3\text{'s} \ldots}} t^{area} q^{dinv}$$

(i.e. read(P) is a shuffle of $1\,2 \ldots \ell \sqcup\!\sqcup\ \ell+1, \ldots, \ell+\lambda_1 \sqcup\!\sqcup \ldots$)



Alternate Model



i	r_i
1	0 $= r_1$
2	1
3	2
4	2
5	1
6	2
7	0
8	0
9	0
10	1
11	1
	10 = area

j	11	10	9	8	7	6	5	4	3	2	1
c_j	2	2	1	2	1	0	0	0	1	1	0

$N = 11$

λ: 9 8 8 7 6 3 3 1 0 0 0

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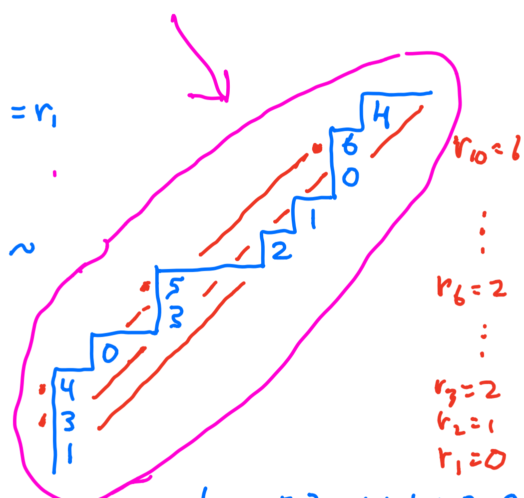


$\text{dinv} = 3+0+1+2+2+3+3+0+0+1 = 15$

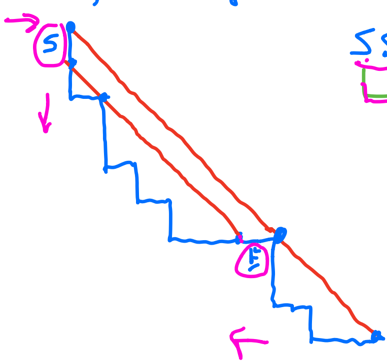
5 0 4

Lemma For any path,

$$\prod_{\substack{1\le i\le N\\ r_i=r_{i-1}+1}} \left(1+\frac{z}{t^{r_i}}\right) = \prod_{\substack{1\le i\le N\\ c_i=c_{i-1}+1}} \left(1+\frac{z}{t^{c_i}}\right)$$

Pf. View a path as a sequence of S, E steps

leftmost S of each SS double step



SSESSESESEEE SSESEE

S → left paren (

E → right paren)

pair

pairs with the rightmost E of a corresponding EE step

Next Change

Def'n Let $[a,b]=\{a,a+1,\dots,b\}$ and $[b]=\{1,\dots,b\}$.
For $\eta\in\mathbb{N}^n$ and $I\subset[n]$ let

$$h_I(\eta)=\left|\{(r<s): r\in I,\ s\notin I,\ \eta_s=\eta_r+1\}\right|$$

where $(r<s)$ denotes of positions (r,s) with $1\le r<s\le n$.

Thm 5.1.1. For $0\le \ell<m\le N$,

ignore 0's

$$\langle z^{N\cdot m}\rangle \sum_{\substack{\lambda\in \mathcal{D}_N \\ P\in L_{N,\ell}(\lambda)}} \prod_{\substack{1<i\le N \\ c_i=c_{i-1}+1}} \left(1+\frac{z}{t^{c_i}}\right) q^{\text{dinv}}\, X^{\text{wt}}$$

ℓ 0's, none in the top row

$$= \sum_{\substack{J\subset[m-1] \\ |J|=\ell}} \; \sum_{\substack{\tau,\,(0,a)\in\mathbb{N}^m \\ |\tau|=N-m}} t^{|a|}\, q^{d((0,a),\tau)+h_J(a)}\, N_{\beta/\alpha}(X;q)$$

where $\beta=(0,a)+(1^m)+\tau$ and $\alpha=(a,0)+\epsilon_J$

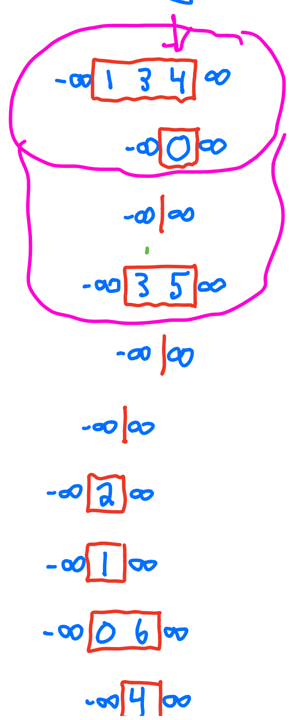
and $N_{\beta/\alpha}$ is defined below

$$(\epsilon_J)_i = \begin{cases} 0 & \text{if } i\notin J \\ 1 & \text{if } i\in J\end{cases}$$

ex. type of object in $N_{\beta/\alpha}$ with

$\beta=(1\,2\,2\,1\,1\,1\,2\,3\,2\,3\,3)$

$\alpha=(1\,1\,0\,0\,0\,1\,2\,1\,2\,2\,0)$



LLT data from parking function above:

$\beta=(1,\, c_2+1,\, c_3+1, \ldots, c_N+1)$

$\alpha=(c_2,\, c_3, \ldots, c_N,\, 0)$

$-\infty \mid \infty$

$\beta_i - \alpha_i$ = # South steps on that line $x = N-i$

ex. $1 - c_2$ south steps on line $x = N-1$

$N-i$