ex. type of object in $N_{\beta/\alpha}$ with

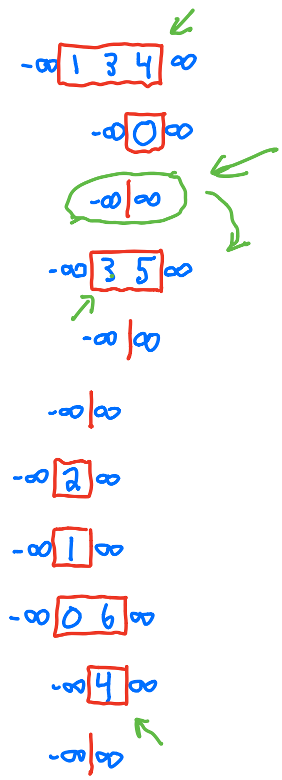
$\beta = (1\,2\,2\,1\,1\,1\,2\,3\,2\,3\,3)$ ←

$\alpha = (1\,1\,0\,0\,0\,1\,2\,1\,2\,2\,0)$ ←

How many triples?

0 : 1
3 : 3
5 : 2
2 : 3
1 : 4
0 : 5
6 : 5
4 : 6

29



3 4
3
$-\infty$ ∞
3

$\alpha_i > \beta_i \Rightarrow 0$

LLT data from parking function above:

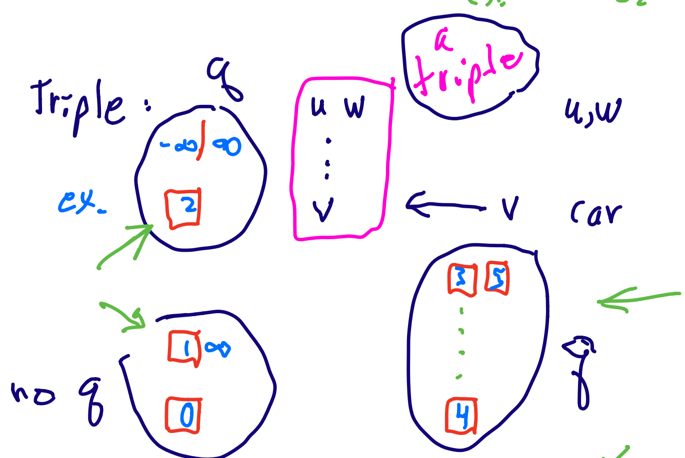
$\beta = (1, c_2+1, c_3+1, \ldots, c_N+1)$

$\alpha = (c_2, c_3, \ldots, c_N, 0)$

LLT data for λ

$\beta_i - \alpha_i$ = # South steps on that line $x = N-i$

ex. $1 - c_2$ south steps on line $x = N-1$



u, w cars or $\pm\infty$

$\in \mathbb{N}$

v car (base square)

$\infty \times \infty$

Increasing triple: $S(u) < S(v) < S(w)$

$S(i)$: entry in square i

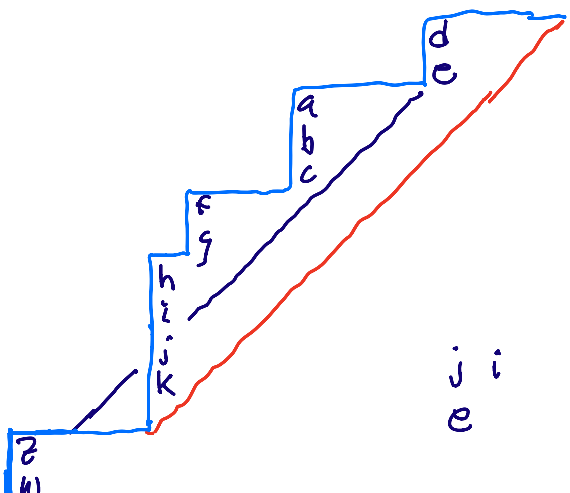
Claim # Increasing Triples = dinv of corresponding partial filling

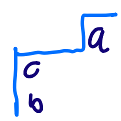
WPF with ℓ 0's
no 0 in top row

Call a triple a 3-car, 2 car, or 1 car triple if $\{u, v, w\}$ contains 3, 2, or 1 cars, respectively

Then the # of 3 car triples with v as a base square equals the # of 1-car triples with v as a base square

mdinv





mdinv = 1

if $c > a$ dinv

if $c \leq a$, then

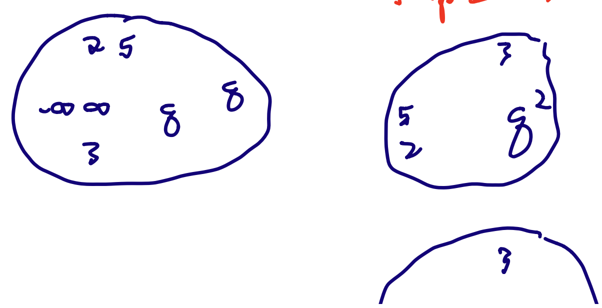
$b < c \leq a$

so a, b dinv

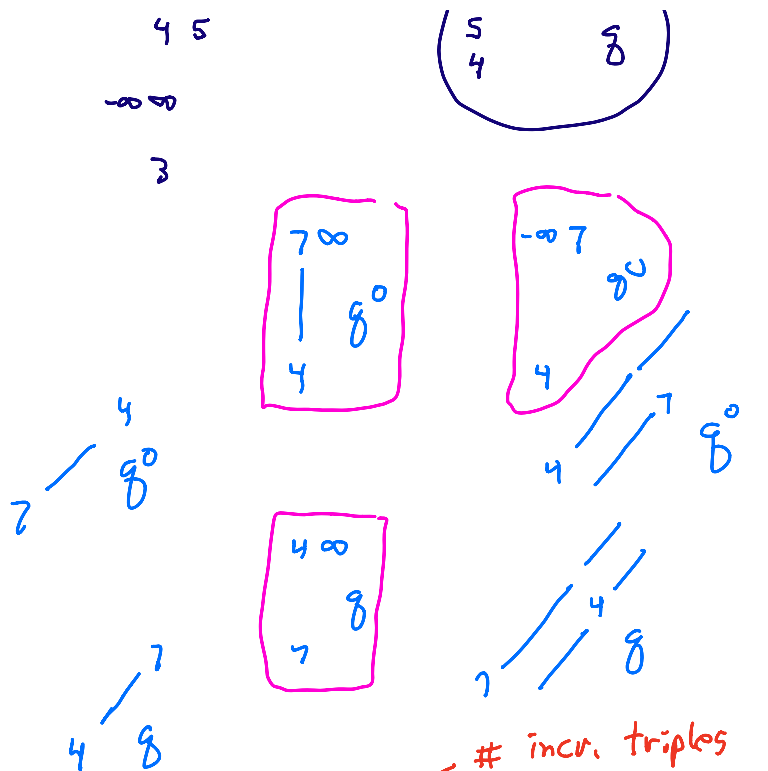
j i 3 car
e triple

$-\infty$ ∞
e

always 8 (increasing triple)



2 car triple



Defn $N_{\beta/\alpha}(X;q) = \sum_{S \in RST(\beta/\alpha,\, \mathbb{Z}_+)} q^{h_{w_0}(S)} X^{wt(S)}$

incr. triples

row strict

Thm 5.1.1

ignore 0's

$$\langle z^{N-m} \rangle \sum_{\substack{\lambda \in \mathcal{D}_N \\ P \in L_{N,\ell}(\lambda)}} \prod_{\substack{1 \le i \le N \\ c_i = c_{i-1}+1}} \left(1 + \frac{z}{t^{c_i}}\right) q^{dinv} X^{wt_+(P)} t^{area}$$

$$= \sum_{\substack{J \subset [m-1] \\ |J| = \ell}} \sum_{\substack{\tau,\, (0,a) \in \mathbb{N}^m \\ |\tau| = N-m}} t^{|a|} q^{d((0,a),\tau) + h_J(a)} N_{\beta/\alpha}(X;q)$$

where $\beta = (0,a) + (1^m) + \tau$ and $\alpha = (a,0) + \epsilon_J$.

Another Defn Give $\tau \in V^m$ $a \in \mathbb{N}^{m-1}$

$\beta_{a\tau}$ = concatenation of sequences
$(1,2,\dots,\tau_1+1)$ and $(a_{i-1}+1, a_{i-1}+2, \cdots, a_{i-1}+\tau_i+1)$
for $2\le i\le N$

---

ex. $a = (1\,3\,0\,0\,1\,2)$ $\quad \tau = (2\,3\;\;1\,1\,0\,2\,2)$

$$(0,a)+(1^m)+\tau = (\underbrace{\_\;\_\;3}_{\tau_1+1}\;\;\underbrace{\_\;\_\;\_\;5}_{\tau_2+1}\;\;\_\;5\;\;\_\;2\;\;1\;\;\_\;\_\;4\;\;\_\;\_\;5)$$

$$\beta_{a\tau} = (1\,2\,3\;\;2\,3\,4\,5\;\;4\,5\;\;1\,2\;\;1\;\;2\,3\,4\;\;3\,4\,5)$$

$$\alpha_{a\tau} = (1\,2\,1\;\;2\,3\,4\,3\;\;4\,0\;\;1\,0\;\;1\;\;2\,3\,2\;\;3\,4\,0)$$

$$(a,0) = (\;1\;\;3\;\;0\;\;0\;\;1\;\;2\;\;0\;)$$

Lemma 5.3.6 For $0\le \ell < m \le N$,

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in\mathcal{D}_N\\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i=c_{i-1}+1}} \left(1+z/t^{c_i}\right) q^{\operatorname{dinv}(P)} X^{\operatorname{wt}_+(P)}$$

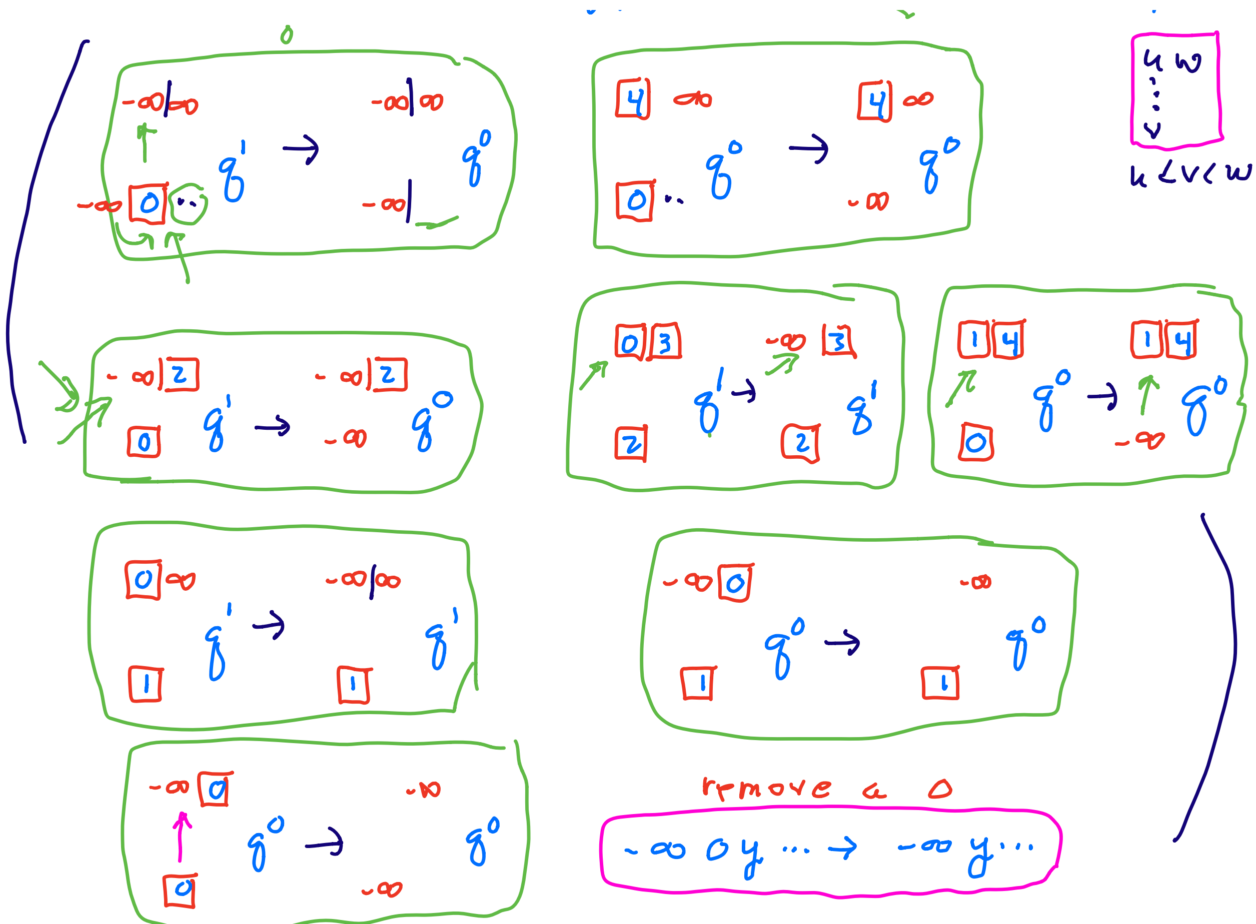
$$= \sum_{\substack{I\subset[N-1]\\ |I|=\ell}} \; \sum_{\substack{\tau,\,(0,a)\in\mathbb{N}^m\\ |\tau|=N-m}} t^{|a|} q^{h_I(\alpha_{a\tau})} N_{\beta_{a\tau}/(\alpha_{a\tau}+\epsilon_I)}(X;q)$$

(*)

0-1 vector, 1's in coordinates $i\in I$

Pf. Remove 0's to rewrite LHS of (*) as

$$\langle z^{N-m}\rangle \sum_{\lambda\in\mathcal{D}_N} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N\\ c_i=c_{i-1}+1}} \left(1+\frac{z}{t^{c_i}}\right) \sum_{\substack{I\subset[N-1]\\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\epsilon_I)}$$



So q power of RST obtained by removing all zeros = q-power of original minus # of $-\infty$ symbols above a zero (and in a row without a zero)

$$h_{w_0}(T) + h_I(\alpha) = h_{w_0}(S)$$

RST with 0's removed

original RST

row above r, without a 0

Here $h_I(\eta) = \left| \{ (r<s) : r \in I,\ s \notin I,\ \eta_s = \eta_r + 1 \} \right|$

$I \subset [N]$

$\alpha = (c_2, c_3, \ldots, c_\mu, 0)$

$\alpha_s = \alpha_r + 1$