Lemma 5.3.6 (continued from BHMPS2b)

For $0 \le \ell < m \le N$,

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in D_N \\ P\in L_{N,\ell}(\lambda)}} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N \\ c_i=c_{i-1}+1}} \left(1+z/t^{c_i}\right) q^{\mathrm{dinv}(P)} x^{\mathrm{wt}_+(P)} \qquad (*)$$

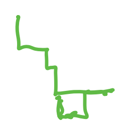
$$= \sum_{\substack{I\subset[N-1] \\ |I|=\ell}} \sum_{\substack{\tau,\,(0,a)\in \mathbb{N}^m \\ |\tau|=N-m}} t^{|a|} q^{h_I(\alpha_{a\tau})} N_{\beta_{a\tau}/(\alpha_{a\tau}+\epsilon_I)}(X;q)$$

0-1 vector, 1's in coordinates $i\in I$

Pf. By previous discussion, LHS of $(*)$

$$=\langle z^{N-m}\rangle \sum_{\lambda\in D_N} t^{|\delta/\lambda|} \prod_{\substack{1<i\le N \\ c_i=c_{i-1}+1}} \left(1+\frac{z}{t^{c_i}}\right) \sum_{\substack{I\subset[N-1] \\ |I|=\ell}} q^{h_I(\alpha)} N_{\beta/(\alpha+\epsilon_I)}$$

where $\beta=(1^N)+(0,c_2,\dots,c_N)$, $\alpha=(c_2,\dots,c_N,0)$ are the LLT data for λ. Note $\vec{c}\in\mathbb{N}^N$ is the vector of column hts for some λ iff $c_1=0$ and $c_s\le c_{s-1}+1$ all $s>1$, in which case $|\delta/\lambda| = |c| = \text{area}$. So LHS of $(*)$ equals



$$\langle z^{N-m}\rangle \sum_{A\subset[N]\setminus\{1\}} \sum_{\substack{c_i=c_{i-1}+1\ \forall i\in A \\ c_i\le c_{i-1}+1\ \forall i\notin A}} t^{|c|-\sum_{i\in A}c_i} z^{|A|} \underbrace{\sum_{\substack{I\subseteq[N-1] \\ |I|=\ell}} N_{\beta/(\alpha+\epsilon_I)} q^{h_I(\alpha)}}_{\text{same as above}}$$

$$= \sum_{\substack{\{1\} \subset J \subset [N] \\ |J| = m \\ (J = [N] \setminus A)}} \sum_{\substack{c_j = c_{j-1}+1 \\ \forall j \,\&\, J}} t^{\sum_{i \in J} c_j} \underbrace{\sum_{\substack{I \subseteq [N-1] \\ |I| = \ell}} N_{\beta/(\alpha+\epsilon_I)} q^{h_I(\alpha)}}_{\text{Same as above}}$$

(Note if $c_j > c_{j-1}+1$ then $\alpha_j > \beta_j \Rightarrow N_{\beta/(\alpha+\epsilon_I)} = 0$)

Next replace the sum over J by a sum over $\{\tau \in \mathbb{N}^m, |\tau| = N-m\}$

Using $J = \{1, \tau_1+2, \tau_1+\tau_2+3, \cdots, \tau_1+\tau_2+\cdots+\tau_{m-1}+m\}$

($\tau_1+\cdots+\tau_{m-1}+m \le N$ so let $\tau_m = N-m-(\tau_1+\cdots+\tau_{m-1})$)

and then for fixed τ (or fixed J) the sum over c can be replaced by a sum over

$$c = (\underbrace{0, 1, 2, \cdots, \tau_1}_{\text{length } \tau_1+1}, \underbrace{a_1, a_1+1, \cdots, a_1+\tau_2}_{\text{length } \tau_2+1}, \cdots, \underbrace{a_{m-1}, a_{m-1}+1, \cdots, a_{m-1}+\tau_m}_{\text{length } \tau_m+1})$$

$a \in \mathbb{N}^{m-1}$, $c \in \mathbb{N}^N$. Note $\sum_{j \in J} c_j = |a| = 0 + a_1 + \cdots + a_{m-1}$

Then $\beta/\alpha = \beta_{a\tau}/\alpha_{a\tau}$ for this encoding of c, since

$\beta/\alpha = (1, c_2+1, c_3+1, \cdots, c_N+1)/(c_1, c_2, \cdots, c_N, 0)$

$= (1, 2, \cdots, \tau_1+1, a_1+1, a_1+2, \cdots, a_1+1+\tau_2, \cdots, a_{m-1}+1, a_{m-1}+2, \cdots, a_{m-1}+1+\tau_m) /$
$(1, 2, \cdots, \tau_1, a_1, a_1+1, \cdots, a_1+\tau_2, \cdots, a_{m-1}, a_{m-1}+1, \cdots, a_{m-1}+\tau_m, 0)$

and recall by definition,

$\beta_{a\tau}$ = concatenation of sequences
$(1, 2, \cdots, \tau_1+1)$ and $(a_{i-1}+1, a_{i-1}+2, \cdots, a_{i-1}+\tau_i+1)$
for $2 \le i \le N$

---

ex. $a = (1\,3\,0\,0\,1\,2)$ $\quad \tau = (2\,3\,\,1\,1\,0\,2\,2)$

$$(0,a)+(1^m)+\tau = (\underbrace{\_\ \_\ 3}_{\tau_1+1}\ \ \underbrace{\_\ \_\ \_\ 5}_{\tau_2+1}\ \ \_\ 5\ \ \_\ 2\ \ 1\ \ \_\ \_\ 4\ \ \_\ \_\ 5)$$

$$\beta_{a\tau} = (1\,2\,3\ \ 2\,3\,4\,5\ \ 4\,5\ \ 1\,2\ \ 1\ \ 2\,3\,4\ \ 3\,4\,5)$$

$$\alpha_{a\tau} = (1\,2\,1\ \ 2\,3\,4\,3\ \ 4\,0\ \ 1\,0\ \ 1\ \ 2\,3\,2\ \ 3\,4\,0)$$

$$(a,0) = (1\ \ 3\ \ 0\ \ 0\ \ 1\ \ 2\ \ 0)$$

Final adjustment to RHS of ✱: Note empty rows can be removed at the cost of a q-factor (ex. $-\infty | \infty$ ← empty row, $\boxed{4}$, q)

We have $\beta_{a\tau}/(\alpha_{a\tau}+\epsilon_I)$ has $|\tau|$ necessarily empty rows. Given $a \in \mathbb{N}^{m-1}$, $\tau \in \mathbb{N}^m$ and $\beta_{a\tau}/\alpha_{a\tau}$, set $j_\uparrow = j + \sum_{x \le j} \tau_x$ for $j \in [m]$ so entry of $\beta_{a\tau}$ in position $j_\uparrow$ is $a_{j-1}+\tau_j+1$ (or τ_1+1 if $j=1$) + entry of $\alpha_{a\tau}$ in position $j_\uparrow$ is a_j (or 0 if $j=m$). For a subset $J \subseteq [m]$, we set $J_\uparrow = \{j_\uparrow : j \in J\}$. In positions $i \notin [m]_\uparrow$, $\beta_{a\tau}$ and $\alpha_{a\tau}$ agree, so row i is empty in $\beta_{a\tau}/\alpha_{a\tau}$. The tuple of row shapes obtained by deleting these empty rows from $\beta_{a\tau}/\alpha_{a\tau}$ is $((0,a)+(1^m)+\tau)/(a,0)$ where row j corresponds to row $j_\uparrow$ of $\beta_{a\tau}/\alpha_{a\tau}$. Note rows $(j-1)\uparrow$ and $j\uparrow$ are separated by τ_j empty rows.

Anna

Lemma 5.3.7 For $J \subset [m]$, $a \in \mathbb{N}^{m-1}$ and $\tau \in \mathbb{N}^m$, let $I = J_\uparrow$. Then

$$N_{\beta_{a\tau}/\alpha_{a_\tau}+\epsilon_I} = q^{d((0,a),\tau) - h'_J(a,\tau)}\, N_{((0,a)+(1^m)+\tau)/((a,0)+\epsilon_J)}$$

where $h'_J(a,\tau) = \left| \{ (j<r) : j \in J, r \in [m], a_j \in [a_{r-1}, a_{r-1}+\tau_j - 1] \} \right|$

with $a_0 = 0$ and

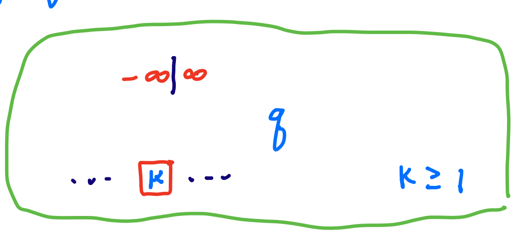
$$d((0,a),\tau) = \sum_{1 \le j < r \le m} \left| [(0,a)_j, (0,a)_j + \tau_j] \cap [(0,a)_r, (0,a)_r + \tau_r - 1] \right|$$

with $[u,v] = \{u, u+1, \dots, v\}$.

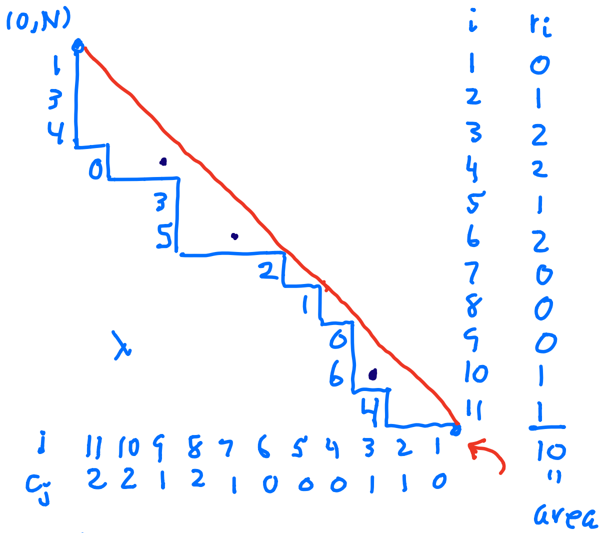
<u>Pf.</u> By removing the $|\tau|$ empty rows of $\beta_{a\tau}/\alpha_{a\tau}$ which are not one of the rows in $\{ j_\uparrow,\ 1 \le j \le m\}$, we are left with $N_{((0,a)+(1^m)+\tau)/(a,0)}$. If we have zeros in a RST of shape $\beta_{a\tau}/\alpha_{a\tau}$ in rows in $J_\uparrow$, $J \subset [m]$, then after removing the $|\tau|$ empty rows we get a RST of shape $((0,a)+(1^m)+\tau)/(a,0)$ with zeros in rows in J. Hence

$$N_{\beta_{a\tau}/\alpha_{a_\tau}+\epsilon_I} = q^F\, N_{((0,a)+(1^m)+\tau)/((a,0)+\epsilon_J)}, \quad (**)$$

where F is the change in h_{w_0} created by removing the $|\tau|$ empty rows from a RST, i.e.

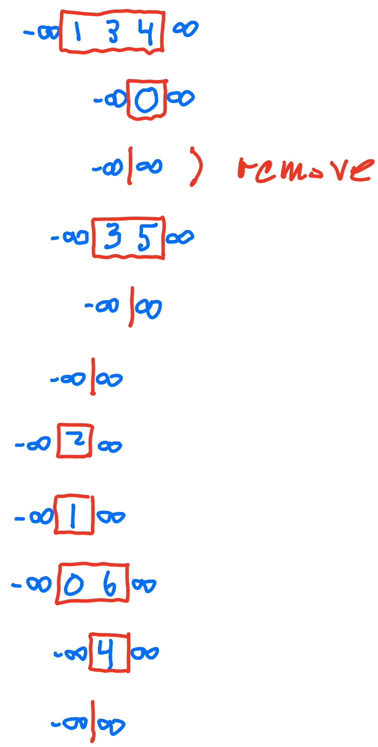
$F = \#$ of 

$\leftarrow$ one of $|\tau|$ removed empty rows



$N = 11$

λ: 98876331000

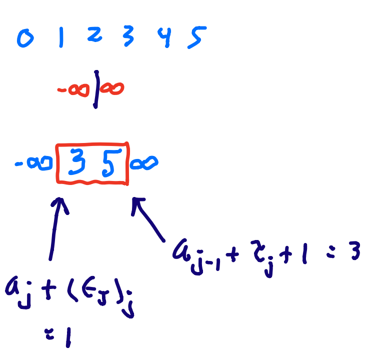


Pf. Set $a_0 = 0$. Note we can assume $a_j + (\epsilon_J)_j \le a_{j-1} + z_j + 1$ $\forall$ $j \in [m]$ else both sides of (**) are zero.

To evaluate F, consider an empty row R of form $(b)/(b)$, $b \in \{a_{r-1}+1, \dots, a_{r-1} + z_r\}$ some $r \in [m]$.

A nonempty lower row j_T of the form $(a_{j-1} + z_j + 1)/(a_j + (\epsilon_J)_j)$ will form an increasing triple with R iff $b \in [a_j + (\epsilon_J)_j + 1, a_{j-1} + z_j + 1]$

ex. $b = 5$, $a_{j-1} + (\epsilon_J)_j = 3$, $a_{j-1} + z_j + 1 = 6$



(2)/(2)

(3)/(1)

Values of b-1 for empty rows (b)/(b)

Hence,

$$F = \sum_{1 \le j < r \le m} \left| [a_j + (\epsilon_J)_j \, , \, a_{j-1} + \tau_j] \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|$$

$$= \sum_{1 \le j < r \le m} \left| [a_j \, , a_{j-1} + \tau_j] \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|$$

$$- \sum_{j \in J} \left| \{a_j\} \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|$$

The sum after the minus sign is $h'_J(a, \tau)$ by defn.

The other sum equals

$$\sum_{1 \le j < r \le m} \Big(\left| [a_j, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right| - \left| [a_{j-1} + \tau_j + 1, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right| \Big). \qquad ***$$

Now since $a_0 = 0 \le a_{r-1}$,

$$\left| [a_{r-1}, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right| = \left| [a_0, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|.$$

Adding $\sum_{1 < j < r} \left| [a_{j-1}, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|$ to both sides we get

$$\sum_{1 \le j < r} \left| [a_j, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right| = \sum_{1 \le j < r} \left| [a_{j-1}, \infty) \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right|$$

Hence *** is unchanged upon replacing $[a_j, \infty)$ with (a_{j-1}, ∞) and is thus equal to

$$\sum_{1 \le j < r \le m} \left(\left| [a_{j-1}, a_{j-1} + \tau_j] \cap [a_{r-1}, a_{r-1} + \tau_r - 1] \right| = d((0, a), \tau) \right.$$

since by defn,

$$d((0,a),\tau) = \sum_{1\le j<r\le m} \left| [(0,a)_j, (0,a)_j+\tau_j] \cap [(0,a)_r, (0,a)_r+\tau_r-1] \right|$$

Theorem 5.1.1

$$\langle z^{N-m}\rangle \sum_{\substack{\lambda\in\mathcal{D}_N \\ P\in L_{N,\ell}(\lambda)}} \prod_{\substack{1\le i\le N \\ c_i=c_{i-1}+1}} \left(1+\frac{z}{t^{c_i}}\right) q^{\operatorname{dinv}} X^{\operatorname{wt}_+(P)} t^{\operatorname{area}}$$

$$= \sum_{\substack{J\subset[m-1] \\ |J|=\ell}} \sum_{\substack{\tau,(0,a)\in\mathbb{N}^m \\ |\tau|=N-m}} t^{|a|} q^{d((0,a),\tau)+h_J(a)} N_{\beta/\alpha}(X;q)$$

where $\beta=(0,a)+(1^m)+\tau$ and $\alpha=(a,0)+\epsilon_J$.

Pf. Consider a summand $t^{|a|} q^{h_I(\alpha_{a\tau})} N_{\beta_{a\tau}/(\alpha_{a\tau}+\epsilon_I)}$ on RHS of $(*)$ (Lemma 5.3.6) for $I\subset[N-1]$, $a\in\mathbb{N}^{m-1}$, $\tau\in\mathbb{N}^m$.

It vanishes unless $I=J_\uparrow$ for some $J\subset[m-1]$ (else $\alpha_{a\tau}+\epsilon_I>\beta_{a\tau}$ at some coordinate). For $I=J_\uparrow$, by Lemma 5.3.7 we can replace the summand with

$$t^{|a|} q^{d((0,a),\tau)+h_I(\alpha_{a\tau})-h_J'(a,\tau)} N_{((0,a)+(1^m)+\tau)/((a,0)+\epsilon_J)}$$

Thm. 5.1.1 now follows from the following

Claim For $\alpha = \alpha_{a\tau}$,

$$h_I(a) = h_J'(a,\tau) + h_J(a)$$

$\boxed{J \subset [m],\ I = J_\uparrow}$

Pf. Recall $N = m_\uparrow$ (since $m_\uparrow = m + \sum_{x \le m} \tau_x$) and note

$$[N]\setminus I = N \setminus [m]_\uparrow \cup [m]_\uparrow \setminus I = N \setminus [m]_\uparrow \cup ([m]\setminus J)_\uparrow$$

(since $([m]\setminus J)_\uparrow = [m]_\uparrow \setminus J_\uparrow$). Thus

$$h_I(\alpha) \overset{\text{by defn}}{=} \left|\{(x<y) : x \in I, y \in [N]\setminus I, \alpha_y = \alpha_x + 1\}\right| = |S_1| + |S_2|$$

where $S_1 = \{(x<y) : x \in J_\uparrow, y \in [N]\setminus[m]_\uparrow, \alpha_y = \alpha_x + 1\}$

$S_2 = \{(x<y) : x \in J_\uparrow, y \in ([m]\setminus J)_\uparrow, \alpha_y = \alpha_x + 1\}$.

Note $\alpha_{m_\uparrow} = 0$ implies $(x, m_\uparrow) \notin S_2$ for any $x < [m]_\uparrow$.

Now $a_u = \alpha_{u_\uparrow}$ $\forall u \in [m-1]$ so

$$h_J(a) = |S_2| = \left|\{(j<r) : j \in J, r \in [m-1]\setminus J, a_r = a_j + 1\}\right|$$

Furthermore, $\{(j<r) : j \in J, r \in [m], a_{r-1}+1 \le a_j + 1 \le a_{r-1} + \tau_r\}$ and S_1 are equinumerous since a pair $(j<r)$ in the 1st set corresponds to the pair $(j_\uparrow < y)$ in S_1, where y is the unique row index in the range $(r-1)_\uparrow < y < r_\uparrow$ such that $\alpha_y = \alpha_{j_\uparrow} + 1 = a_j + 1$.

$a_{r-1} \le a_j$
$j < r$

$\beta_{a\tau}$ = concatenation of sequences $(1, 2, \ldots, \tau_1 + 1)$ and $(a_{i-1}+1, a_{i-1}+2, \cdots, a_{i-1} + \tau_i + 1)$ for $2 \le i \le N$

$a_j + 1 \le a_{r-1} + \tau_r$

ex. $a = (1\,3\,0\,0\,1\,2)$ $\quad \tau = (2\,3\,1\,1\,0\,2\,2)$

$$(0,a) + (1^m) + \tau = (\overbrace{\_\ \_\ 3}^{\tau_1+1}\quad \overbrace{\_\ \_\ \_\ 5}^{\tau_2+1}\quad \_\ 5\quad \_\ 2\quad 1\quad \_\ \_\ 4\quad \_\ \_\ 5)$$

$$\beta_{a\tau} = (1\,2\,3\quad 2\,3\,4\,5\quad 4\,5\quad 1\,2\quad 1\quad 2\,3\,4\quad 3\,4\,5)$$

$$\alpha_{a\tau} = (1\,2\,1\quad 2\,3\,4\,3\quad 4\,0\quad 1\,0\quad 1\quad 2\,3\,2\quad 3\,4\,0)$$

$$(a,0) = (1\quad 3\quad 0\quad 0\quad 1\quad 2\quad 0)$$

$j\uparrow$ $\quad$ column $\hat{2}$ $\quad$ $r\text{-}1\uparrow$ $\quad y$ $\quad r\uparrow$

a_{r-1} $\quad a_{r-1}+1$ $\quad a_{r-1}+\tau_r$

$\hat{1} = 1+\tau_1$

$\hat{2} = 2+\tau_1+\tau_2$

$\vdots$

$\alpha_{j\uparrow}+1 = \beta_{j\uparrow+1} = a_j+1$

$\beta = \beta_{a\tau}$

$\alpha = \alpha_{a\tau}$

$(r\text{-}1)\uparrow < y < r\uparrow$

$\alpha_y = \alpha_{j\uparrow}+1$

ex. $j=1$, $r-1=5$

$a_5+1 \le a_1+1 \le a_5+\tau_6$

i.e. $2 \le 2 \le 1+2$ ✓

$(0,a)+(1^m)+\tau\,/\,(a,0)$ $\qquad\qquad$ $\beta_{a\tau}/\alpha_{a\tau}$