The Cauchy Identity

Recall

$$E_\lambda^\sigma(x;q) = q^{|\mathrm{Inv}(\sigma^{-1}) \cap \mathrm{Inv}(\lambda+\epsilon\rho)|} \boxed{T_{\sigma^{-1}}^{-1}} E_{\sigma^{-1}(\lambda)}(x;q) \qquad (1)$$

id

$$F_\lambda^\sigma(x;q) = \overline{E_{-\lambda}^{\sigma w_0}(x;q)}$$

$$E_\lambda^\sigma = \begin{cases} q^{-I(\lambda_i \le \lambda_{i+1})} \, T_i \, E_{s_i\lambda}^{s_i\sigma} & s_i\sigma > \sigma \\ q^{I(\lambda_i \ge \lambda_{i+1})} \, T_i^{-1} \, E_{s_i\lambda}^{s_i\sigma} & s_i\sigma < \sigma \end{cases} \qquad (2)$$

where $I(P) = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$

$$\boxed{E_\lambda^\sigma = x^\lambda \quad \forall \sigma \text{ if } \lambda = \lambda_+}$$

Prop. 4.3.2 of BHMPS1 $\forall \sigma \in S_\ell$, the $E_\lambda^\sigma(x;q)$ and $\overline{F_\lambda^\sigma(x;q)}$ are dual bases of $k[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}]$ with respect to the inner product defined by

$$\langle f, g \rangle_q = \langle x^0 \rangle \, f g \boxed{\prod_{i<j} \frac{1 - x_i/x_j}{1 - q^{-1} x_i/x_j}}, \qquad (*)$$

i.e. $\langle E_\lambda^\sigma, \overline{F_\mu^\sigma} \rangle_q = \delta_{\lambda\mu} \quad \forall \lambda, \mu \in \mathbb{Z}^\ell$ and $\sigma \in S_\ell$

Lemma 4.3.3 of BHMPS1. With

$T_i = q s_i + (1-q) \dfrac{1}{\qquad} (s_i - 1)$,

$1 - x_{i+1}/x_i$

T_i is self-adjoint w.r.t. $\langle -, - \rangle_q$

Pf. It suffices to show $T_i - q$ is self adjoint since

$$\langle (T_i - q) f, g \rangle = \langle f, (T_i - q) g \rangle \Rightarrow \langle T_i f, g \rangle = \langle f, T_i g \rangle$$

Now $\boxed{T_i - q = q \, \frac{1 - q^{-1} x_i/x_{i+1}}{1 - x_i/x_{i+1}} (s_i - 1)}$ since

$$T_i - q = (s_i - 1)\left[(1-q) \frac{x_i}{x_i - x_{i+1}} + q \right]$$

$$= (s_i - 1)\left[q\left(1 - \frac{x_i}{x_i - x_{i+1}} \right) + \frac{x_i}{x_i - x_{i+1}} \right]$$

$$= (s_i - 1)\left[q\left(1 + \frac{x_i/x_{i+1}}{1 - x_i/x_{i+1}} \right) - \frac{x_i/x_{i+1}}{1 - x_i/x_{i+1}} \right]$$

$$= \frac{(s_i - 1)}{1 - x_i/x_{i+1}} \left[q - x_i/x_{i+1} \right].$$

thus

$$\langle (T_i - q) f, g \rangle_q = q \langle x^0 \rangle \left[\left(\frac{1 - q^{-1} x_i/x_{i+1}}{1 - x_i/x_{i+1}} (s_i - 1) \right) f \right] g \prod_{i<j} \frac{1 - x_i/x_j}{1 - q^{-1} x_i/x_j}$$

$$= q \langle x^0 \rangle \left(s_i(f) g - f g \right) \prod_{\substack{j<k \\ (j,k) \neq (i,i+1)}} \frac{1 - x_j/x_k}{1 - q^{-1} x_j/x_k}. \qquad (**)$$

We want this to be invariant under $s_i(f) g \leftrightarrow f s_i(g)$ i.e. symmetric in f, g.

Let Δ = product factor in (**). Note Δ is symmetric in x_i, x_{i+1}. For any $\varphi(x_1, \dots, x_\ell)$,
$\langle x^0 \rangle \varphi(x_1, \dots, x_\ell) = \langle x^0 \rangle s_i \varphi(x_1, \dots, x_\ell)$ so

$\langle x^0 \rangle s_i(f) g \, \Delta = \langle x^0 \rangle f \, s_i(g) \, \Delta$ so (**) is invariant under $s_i(f) g \leftrightarrow f \, s_i(g)$ $\square$

Pf. of Prop 4.3.2

$\langle E^\sigma_\lambda, \overline{F^\sigma_\mu} \rangle_q = \delta_{\lambda\mu}$ $\forall \lambda, \mu \in \mathbb{Z}^\ell$ and $\sigma \in S_\ell$ is equivalent to

$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = \delta_{\lambda\mu}$ $\forall \lambda, \mu \in \mathbb{Z}^\ell$ and $\sigma \in S_\ell$

$s_i \sigma > \sigma$
$s_i \sigma w_0 < \sigma w_0$

By (2), for every i either

$$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = q^e \langle T_i E^{s_i \sigma}_{s_i \lambda}, T_i^{-1} E^{s_i \sigma w_0}_{-s_i \mu} \rangle_q \quad \text{or}$$

$$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = q^e \langle T_i^{-1} E^{s_i \sigma}_{s_i \lambda}, T_i E^{s_i \sigma w_0}_{-s_i \mu} \rangle_q,$$ depending

on whether $s_i \sigma > \sigma$ or $s_i \sigma < \sigma$, for some e.

(since i occurs before $i+1$ in σ iff $i+1$ occurs before i in σw_0)

$$\begin{matrix} 1 & \dots & i & i+1 & \dots & n \\ \sigma_1 & \dots & \sigma_i & \sigma_{i+1} & \dots & \sigma_n \end{matrix} \cdot \begin{matrix} 1 & 2 & & i & i+1 & \dots & n \\ n & n-1 & & n-i+1 & n-i & \dots & 1 \end{matrix}$$

$$= \begin{matrix} 1 & 2 & \dots & n-i & n-i+1 & \dots & n \end{matrix}$$

$$\sigma_n \cdots \sigma_{i+1} \quad \sigma_i \cdots \sigma_1$$

$$n-i+1 \to i \to \sigma_i$$

$$n-i \to i+1 \to \sigma_{i+1}$$

Since T_i is self-adjoint, we get

$$\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{s_i\sigma}_{s_i\lambda}, E^{s_i\sigma w_0}_{-s_i\mu}\rangle_q \quad \text{in either case.}$$

Repeating gives

$$\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{v\sigma}_{v\lambda}, E^{v\sigma w_0}_{-v\mu}\rangle_q \qquad \forall\ \lambda,\mu \in \mathbb{Z}^\ell,\ \sigma, v \in S_\ell \qquad (**)$$

with $\boxed{q^e = 1 \text{ if } \lambda = \mu,}$

Choose $v \in S_\ell$ so $\mu_- = v(\mu)$ is antidominant. Then (**) gives

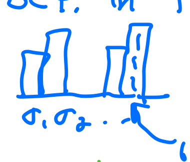
$$\langle E^{\sigma}_{\lambda}, E^{\sigma w_0}_{-\mu}\rangle_q = q^e \langle E^{v\sigma}_{v\lambda}, E^{v\sigma w_0}_{-\mu_-}\rangle_q = q^e \langle E^{v\sigma}_{v\lambda}, x^{-(\mu_-)}\rangle_q$$

$$(*) \quad = q^e \langle x^{\mu_-}, \Delta(x;q)\, E^{v\sigma}_{v\lambda} \rangle$$

where $\Delta(x;q)$ is the product factor in (*). Let supp(f) denote the set of weights ν : x^ν occurs with nonzero coef. in f.

Since $\mathrm{supp}(\Delta(x;q)) = Q_+$ and

$\mathrm{supp}(E^{v\sigma}_{v\lambda}) \subseteq \mathrm{conv}(S_\ell \cdot \lambda)$ $\leftarrow$



$$\Delta(x;q) = \prod_{i<j} \frac{1 - x_i/x_j}{1 - q^{-1}x_i/x_j} = \prod_{i<j}\left(1 + (q^{-1}-1)\frac{x_i}{x_j} + (q^{-2}-q^{-1})\left(\frac{x_i}{x_j}\right)^2 + \cdots\right)$$

then if $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0$, then $(\mu_- - Q_+) \cap \mathrm{conv}(S_\ell \cdot \lambda) \neq 0$

and so $\mu_- - \lambda_- \in Q_+$. Since $w_0(Q_+) = -Q_+$, this is equivalent to $\lambda_+ \geq \mu_+$.

1 2 3 4

0 1 2 3

$$\langle f, g\rangle_q = \langle x^0\rangle \, f \, g \prod_{i<j} \frac{1 - x_i/x_j}{1 - q^{-1} x_i/x_j} \quad , \qquad (*)$$

By symmetry, exchanging λ with $-\mu$ and σ with σw_0, if $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0$ then we also get $(-\lambda)_- - (-\mu)_- \in Q_+$, hence $\lambda_+ - \mu_+ \in -Q_+$, or $\lambda_+ \leq \mu_+$

ex. If $\underbrace{(-3,-3,-1,0)}_{-\lambda_-} - \underbrace{(-2,-2,-2,-1)}_{-\mu_-} \in Q_+$

then $(3,3,1,0) - (2,2,2,1) \in -Q_+$

Hence $\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu}\rangle_q \neq 0 \Rightarrow \lambda_+ = \mu_+$ so λ, μ in same S_ℓ - orbit. In this case,

$(\mu_- - Q_+) \cap \mathrm{conv}(S_\ell \cdot \lambda) = \{\mu_-\}$. Furthermore, if $\lambda \neq \mu$, then $v\lambda \neq \mu_-$ (recall $\mu_- = v(\mu)$)

By

Corollary 4.3.1 $\forall \lambda, \sigma$

$$E^\sigma_\lambda(x;q) = x^\lambda + \sum_{\mu < \lambda} c_\mu x^\mu$$

we have $(\mu_- - Q_+) \cap \mathrm{supp}(E^{v\sigma}_{v\lambda}) = \phi$, hence

$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\mu} \rangle_q = 0$.

If $\lambda = \mu$, then $q^e \langle x^{\mu_-} \rangle \Delta E^{v\sigma}_{v\lambda}$ from ⊛ above

reduces to $\langle x^{\mu_-} \rangle \Delta E^{v\sigma}_{\mu_-}$. Since $\mathrm{supp}(\Delta) = Q_+$,

and $\mathrm{supp}(E^{v\sigma}_{\mu_-}) \subset \mu_- - Q_+$, only the constant term

of Δ and the x^{μ_-} term of $E^{v\sigma}_{\mu_-}$ contribute to

the coef of x^{μ_-} in $\Delta E^{v\sigma}_{\mu_-}$, and we have

$\langle x^{\mu_-} \rangle E^{v\sigma}_{\mu_-} = 1$ by Cor. 4.3.1 above. Hence,

$\langle E^\sigma_\lambda, E^{\sigma w_0}_{-\lambda} \rangle_q = 1$. $\square$