The Cauchy Identity

Thm 5.1.1 For any $\sigma \in S_\ell$, the E^σ_λ & F^σ_λ satisfy

$$\boxed{\frac{\prod_{i<j}(1-qtx_iy_j)}{\prod_{i\le j}(1-tx_iy_j)}} = \sum_{a\in\mathbb{Z}^\ell_{\ge 0}} t^{|a|} E^\sigma_a(x_1,\dots,x_\ell;q^{-1})\, F^\sigma_a(y_1,\dots,y_\ell;q). \quad (*)$$

Pf. Let $Z(x,y;q,t)$ denote the LHS of $(*)$. Since the E^σ_a and F^σ_a form bases for $\mathbb{Q}(q)[x]$, we need to show

$$\langle F^\sigma_a(y^{-1};q^{-1})\rangle\, Z(x,y^{-1};q^{-1},t) = t^{|a|} E^\sigma_a(x;q)$$

Lemma For any $f(y) \in \mathbb{Q}(q)[y_1,\dots,y_\ell]$,

$$f(tx) = \langle y^0\rangle\, f(y)\, \frac{\prod_{i<j}(1-q^{-1}tx_i/y_j)}{\prod_{i\le j}(1-tx_i/y_j)} \prod_{i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j} \quad (**)$$

Pf. The only factor in $(**)$ involving negative powers of y_1 is $\frac{1}{1-x_1/y_1}$. The rest is a power series in y_1.

For any power series $g(y_1)$,

$$\langle y_1^0\rangle \frac{g(y_1)}{1-t\frac{x_1}{y_1}} = g(tx_1)$$

since if $g(x) = \sum_0^\infty a_k x^k$,

$$\langle y_1^0\rangle \frac{g(y_1)}{} = \sum^\infty a_k (tx_1)^k = g(tx_1).$$

So $\langle y_1^0 y_2^0 \cdots y_\ell^0\rangle$ on right in (**) can be obtained by letting $y_1 = tx_1$ and taking $\langle y_2^0 \cdots y_\ell^0\rangle$ in whats left, i.e.

part A

$$\langle y^0\rangle f(y)\, \frac{\prod_{i<j}\left(1-q^{-1}tx_i/y_j\right)}{\prod_{\substack{i\le j\\ j\ne 1}}\left(1-tx_i/y_j\right)}\ \prod_{i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j}$$

$y_1 = tx_1$

this part cancels $i=1$ terms from Part A

$$\prod_{2\le i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j}\ \prod_{j=2}^{\ell}\frac{1-tx_1/y_j}{1-q^{-1}tx_1/y_j}$$

$$= \langle y_2^0 \cdots y_\ell^0\rangle f(tx_1, y_2, \dots, y_\ell)\, \frac{\prod_{2\le i<j}\left(1-q^{-1}tx_i/y_j\right)}{\prod_{2\le i\le j}\left(1-tx_i/y_j\right)}\ \prod_{2\le i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j}$$

and (**) follows by induction. □

Letting $f(y) = E_a^\sigma(y;q)$ in (**) gives

$$E_a^\sigma(tx;q) = \langle y^0\rangle E_a^\sigma(y;q)\, \frac{\prod_{i<j}\left(1-q^{-1}tx_i/y_j\right)}{\prod_{i\le j}\left(1-tx_i/y_j\right)}\ \prod_{i<j}\frac{1-y_i/y_j}{1-q^{-1}y_i/y_j}$$

Recall:

Prop. 4.3.2 of BHMPS1 $\quad \forall\, \sigma\in S_\ell$, the $E_\lambda^\sigma(x;q)$ and

$F^{\sigma}_{\lambda}(x;q)$ are dual bases of $K[x_1^{\pm 1},\dots,x_\ell^{\pm 1}]$ with respect to the inner product defined by

$$\underline{\langle f,g\rangle_q = \langle x^0\rangle\, f\, g \prod_{i<j}\frac{1-x_i/x_j}{1-q^{-1}x_i/x_j}}\,, \qquad (*)$$

i.e. $\langle E^{\sigma}_{\lambda}, \overline{F^{\sigma}_{\mu}}\rangle_q = \delta_{\lambda\mu}$ $\forall\, \lambda,\mu\in\mathbb{Z}^{\ell}$ and $\sigma\in S_\ell$.

Now $\langle F^{\sigma}_{a}(y^{-1};q^{-1})\rangle\, \underline{Z(x,y^{-1};q^{-1},t)} = t^{|a|}E^{\sigma}_{a}(x;q)$:

is equivalent to

$$\langle F^{\sigma}_{a}(y^{-1};q^{-1})\rangle\, \underbrace{\frac{\prod_{i<j}(1-q^{-1}t\,x_i/y_j)}{\prod_{i\le j}(1-t\,x_i/y_j)}}_{Z(x,y^{-1};q^{-1},t)} = t^{|a|}E^{\sigma}_{a}(x;q)$$

$$\| $$

$$= \sum_a F^{\sigma}_{a}(y^{-1};q^{-1})\, t^{|a|}\, E^{\sigma}_{a}(x;q)$$

plug in

$$\prod (1-q^{-1}t\,x_i/y_j)$$

$$E_b^{\sigma}(tx;q) = \langle y^0 \rangle E_b^{\sigma}(y;q) \frac{\cdots}{\prod_{i\le j}(1 - t x_i/y_j)} \prod_{i<j} \frac{1 - y_i/y_j}{1 - q^{-1} y_i/y_j}$$

$$E_b^{\sigma}(tx;q) = \langle y^0 \rangle E_b^{\sigma}(y;q) \sum_{a} F_a^{\sigma}(y^{-1};q^{-1})\, t^{|a|} E_a^{\sigma}(x;q) \prod_{i<j} \frac{1 - y_i/y_j}{1 - q^{-1} y_i/y_j}$$

$a \in \mathbb{Z}^{\ell}_{\ge 0}$

$$= \sum_{a} E_a^{\sigma}(x;q)\, t^{|a|} \langle y^0 \rangle\, E_b^{\sigma}(y;q)\, \overline{F_a^{\sigma}(y;q)} \prod_{i<j} \frac{1 - y_i/y_j}{1 - q^{-1} y_i/y_j}$$

$$= \sum_{a} E_a^{\sigma}(x;q)\, t^{|a|} \left\langle E_b^{\sigma}(y;q), \overline{F_a^{\sigma}(y;q)} \right\rangle_q = E_b^{\sigma}(x;q)\, t^{|b|}$$

by Prop. 4.3.2.

$$= E_b^{\sigma}(tx;q) \qquad \square$$

Pieri Rules

Lemma 4.5.1 from BHMPS1 says for all $\sigma \in S_m$, $k \in \mathbb{Z}_{+}$,

$$e_k E_a^{\sigma^{-1}}(x;q) = \sum_{\substack{I \subseteq [m] \\ |I| = k}} q^{-\iota_I(a,\sigma)} E^{\sigma^{-1}}_{a+\epsilon_I}(x;q)$$

Lemma 6.3.3 For any $a, \alpha \in \mathbb{N}^m$ and any symmetric $f \in \mathbb{Z}[q^{\pm 1}][x_1, \dots, x_m]^{S_m}$,

$$\langle E^{w_0}_{w_0 a}(x;q^{-1}) \rangle f(x) E^{w_0}_{w_0\alpha}(x;q^{-1}) = \overline{\langle E^{id}_{\alpha}(x;q) \rangle} f(x) \overline{E^{id}_a(x;q)}.$$

Pf. By linearity it suffices to prove it for $f = e_\lambda$.

Taking $f = e_{\lambda_1} \cdots e_{\lambda_k}$ on the left and $f = e_{\lambda_k} \cdots e_{\lambda_1}$ on the right,

$$\langle E^{w_0}_{w_0 a}(x;q^{-1})\rangle\, e_{\lambda_1}\cdots e_{\lambda_k}\, E^{w_0}_{w_0\alpha}(x;q^{-1}) = \overline{\langle E^{id}_{\alpha}(x;q)\rangle}\, e_{\lambda_k}\cdots e_{\lambda_1}\, \overline{E^{id}_{a}(x;q)} \qquad (***)$$

we can reduce to the case $f = e_\ell(x)$?

(***) can be deduced from Lemma 4.5.1 above. □

<u>Lemma 6.3.4</u> For $n \in \mathbb{N}^m$, $k \in \mathbb{N}$

$$h_k\, F^{w_0}_{n}(x;q) = \sum_{\substack{\tau \in \mathbb{N}^m \\ |\tau| = k}} q^{d(n,\tau)}\, F^{w_0}_{n+\tau}(x;q)$$

<u>Pf.</u> By defn,

$$\langle \chi_\lambda \rangle \mathcal{L}^{v}_{\beta/\alpha}(x;q^{-1}) = \langle E^{v}_{\beta} \rangle \chi_\lambda E^{v}_{\alpha}.$$

Letting $\lambda = (k)$ gives

$$\langle h_k \rangle \mathcal{L}^{w_0}_{w_0(a/\alpha)}(x;q) = \langle E^{w_0}_{w_0 a}(x;q^{-1}) \rangle h_k(x) E^{w_0}_{w_0 \alpha}(x;q^{-1}).$$

Lemma 6.3.3 then implies

$$\langle h_k \rangle \mathcal{L}^{w_0}_{w_0(a/\alpha)}(x;q) = \langle \overline{E^{id}_{\alpha}(x;q)} \rangle h_k(x) \overline{E^{id}_{a}(x;q)}.$$

$$= \langle F^{w_0}_{-\alpha}(x;q) \rangle h_k(x) F^{w_0}_{-a}(x;q)$$

The result now follows from the combinatorial expression for $\mathcal{L}^{w_0}_{w_0(a/\alpha)}(x;q)_{pol}$

$$\left(\text{i.e. } \mathcal{L}^{w_0}_{\beta/\alpha}(x;q)_{pol} = \underbrace{\sum_{T \in SSYT(\beta/\alpha)} q^{h'_{w_0}(T)} x^{wt(T)}}_{\text{coef of } h_k \text{ in here} = \text{coef of } x_1^k}\right) \qquad \square$$

coef $x_1^k = \langle -, h_k \rangle$