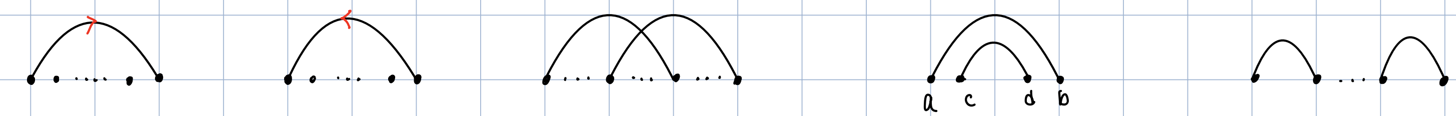


forward direction　　reverse direction　　crossing arcs　　arc (c,d) is nested by arc (a,b)

$\underbrace{\hspace{6em}}_{\text{non-crossing arcs}}$

Given $w \in S_n$, we associate a directed perfect matching $M(w)$ as follows:

(I) n is even: Set $w_0 = 0$, $w_{n+1} = n+1$

Arc 1: From w_n to w_{n+1}

Arc 2: From w_{n-2} to w_{n-1}

$\vdots$

Arc $\frac{n}{2}$: From w_2 to w_3

Arc $\frac{n}{2}+1$: From w_0 to w_1

$\underbrace{\boxed{0\ w_1}}_{\text{arc } \frac{n}{2}+1}\ \underbrace{\boxed{w_2\ w_3}}_{\text{arc } \frac{n}{2}}\ \boxed{w_4} \cdots \underbrace{\boxed{w_n\ w_{n+1}}}_{\cdots\ \text{arc } 1}$

(II) n is odd: Set $w_0 = 0$,

Arc 1: From w_{n-1} to w_n

Arc 2: From w_{n-3} to w_{n-2}

$\vdots$

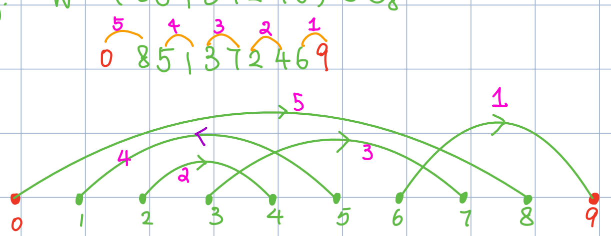
Arc $\frac{n-1}{2}$: From w_2 to w_3

Arc $\frac{n+1}{2}$: From w_0 to w_1

$\underbrace{\boxed{0\ w_1}}_{\text{arc } \frac{n+1}{2}}\ \underbrace{\boxed{w_2\ w_3}}_{\text{arc } \frac{n-1}{2}}\ \underbrace{\boxed{w_4}}_{\text{arc } \frac{n-3}{2}} \cdots \underbrace{\boxed{w_{n-1}\ w_n}}_{\cdots\ \text{arc } 1}$

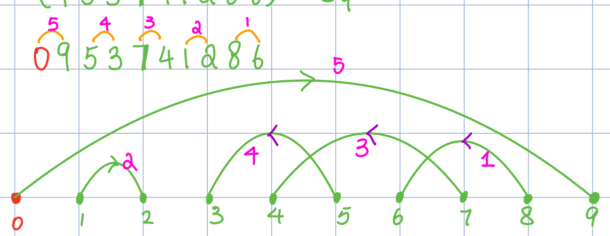
(Pair w_{2i} and w_{2i+1} to form an arc
and index the arcs from right to left.)

e.g. $w = (8\,5\,1\,3\,7\,2\,4\,6) \in S_8$



$M(85137246)$

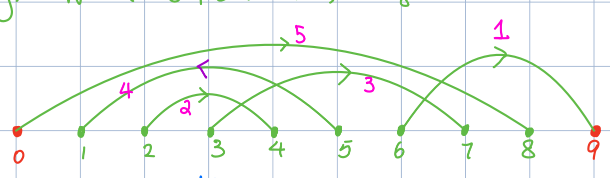
$w = (9\,5\,3\,7\,4\,1\,2\,8\,6) \in S_9$



$M(953741286)$

Define $k(w) := \min_k \{(\text{arc } k, \text{arc } k+1) \text{ of } M(w) \text{ is noncrossing}\}$ if $M(w)$ has at least one non-crossing arcs (arc k, arc $k+1$)

e.g. $w = (8\,5\,1\,3\,7\,2\,4\,6) \in S_8$



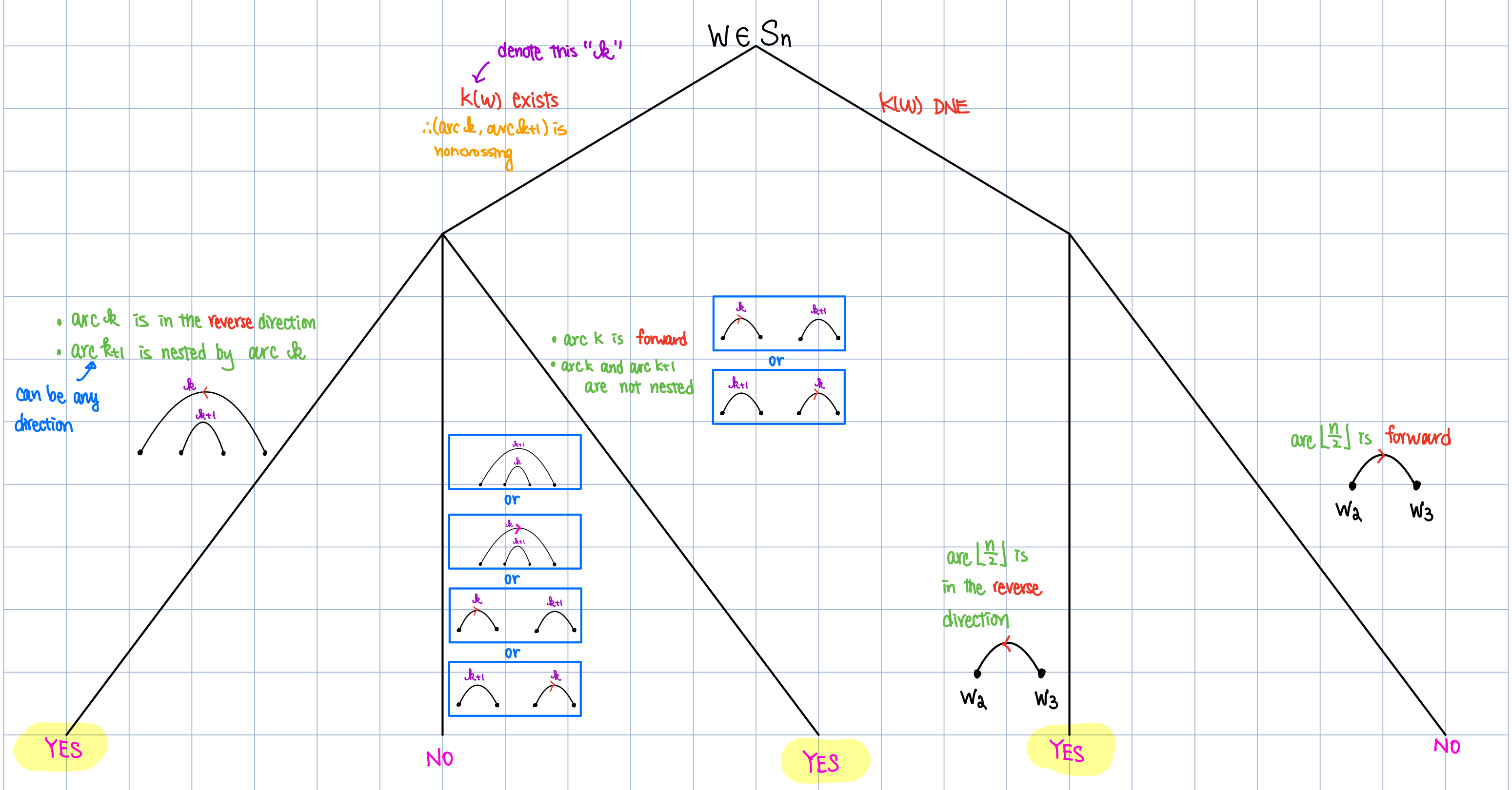
$M(85137246)$

$k(w) = \min\{1, 4\} = 1$

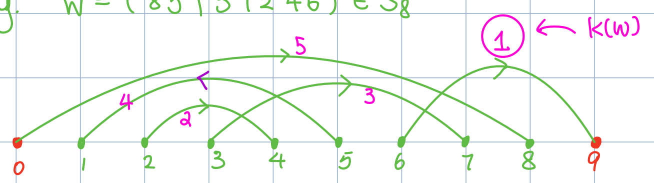
arc 1 and arc 2 are noncrossing　　arc 4 and arc 5 are noncrossing

(but (arc 2, arc 3), (arc 3, arc 4) are both crossing)

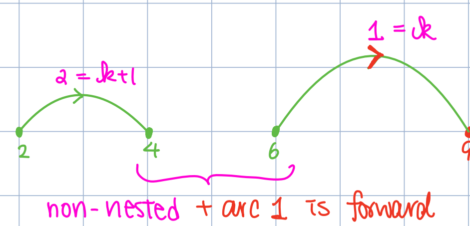
Determine whether $w \in S_n$ is a Butler permutation :



e.g. $w = (8\ 5\ 1\ 3\ 7\ 2\ 4\ 6) \in S_8$

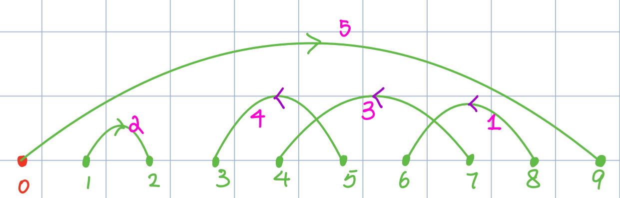


$k(w) = 1$

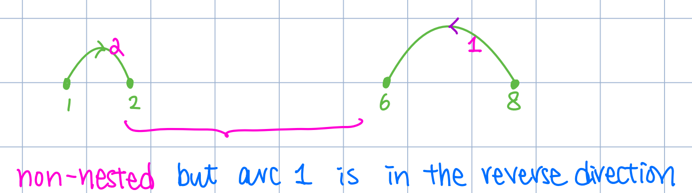


$\Rightarrow$ w is a Butler permutation

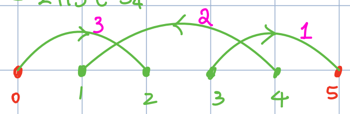
e.g. $w = (9\ 5\ 3\ 7\ 4\ 1\ 2\ 8\ 6) \in S_9$



$k(w) = 1$

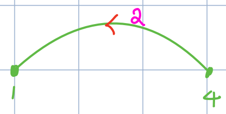


$\Rightarrow$ w is NOT a Butler permutation e.g. $W = 2413 \in S_4$



$K(W)$ DNE

$2\boxed{41}3$ ← $w_2 w_3$



is in reversed direction. $\therefore$ W is a Butler permutation.

Notation: $\mathcal{B}_n = \{ w \in S_n : w \text{ is a Butler permutation} \}$.

Lemma: Let $w \in S_n$ ($n \geq 3$) s.t. (arc 1, arc 2) is crossing in $M(w)$. Then

$$w \text{ is a Butler permutation iff } \begin{cases} std(w_{[n-2]}) \text{ is a Butler permutation} & \text{if } n \text{ is odd} \\ std(w_{[n-1]}) \text{ is a Butler permutation} & \text{if } n \text{ is even} \end{cases}.$$

Proof: If (arc 1, arc 2) is crossing, then $k(w) \neq 1$ and determining whether w is Butler is the same as looking the subgraph of $M(w)$ with arc 1 removed.

$\therefore$ Remove vertices $n+1$ and w_n if n is even ($\because$ we only remove w_n from w)
Remove vertices w_n and w_{n-1} if n is odd ($\because$ remove w_n and w_{n-1} from w)

Lemma: Let m, k be positive integers. Let $A \subseteq [k+m]$ s.t. $|A| = m$. Then for any $\nu \in S_m$, we have

$$\sum_{\substack{w \in S_{k+m} \\ std(w|_A) = \nu}} F_{iDes(w)} = (h_1)^k F_{iDes(\nu)}$$

e.g. $m = k = 2$, $A = \{1, 3\}$, $\nu = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in S_2$

$$\{ w \in S_4 : std(w_1 w_3) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \} = \Big\{ \underset{iDes=\{1\}}{\begin{pmatrix} 1&2&3&4 \\ 2&3&1&4 \end{pmatrix}}, \underset{iDes=\{1,3\}}{\begin{pmatrix} 1&2&3&4 \\ 2&4&1&3 \end{pmatrix}}, \underset{iDes=\{1,2\}}{\begin{pmatrix} 1&2&3&4 \\ 3&2&1&4 \end{pmatrix}}, \underset{iDes=\{2\}}{\begin{pmatrix} 1&2&3&4 \\ 3&4&1&2 \end{pmatrix}}, \underset{iDes=\{1,3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&2&1&3 \end{pmatrix}}, \underset{iDes=\{2,3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&3&1&2 \end{pmatrix}}, \underset{iDes=\{2\}}{\begin{pmatrix} 1&2&3&4 \\ 3&1&2&4 \end{pmatrix}}, \underset{iDes=\{1,2\}}{\begin{pmatrix} 1&2&3&4 \\ 3&4&2&1 \end{pmatrix}},$$

$$\underset{iDes=\{3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&1&2&3 \end{pmatrix}}, \underset{iDes=\{1,2,3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&3&2&1 \end{pmatrix}}, \underset{iDes=\{2,3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&1&3&2 \end{pmatrix}}, \underset{iDes=\{1,3\}}{\begin{pmatrix} 1&2&3&4 \\ 4&2&3&1 \end{pmatrix}} \Big\}$$

$$\sum_{\substack{w \in S_4 \\ std(w|_A) = \nu}} F_{iDes(w)} = F_{\{1\}} + 2F_{\{2\}} + F_{\{3\}} + 2F_{\{1,2\}} + 3F_{\{1,3\}} + 2F_{\{2,3\}} + F_{\{1,2,3\}}$$

$$= h_1^2 \cdot F_{\{2\}} = h_1^2 \cdot F_{iDes(\nu)}.$$