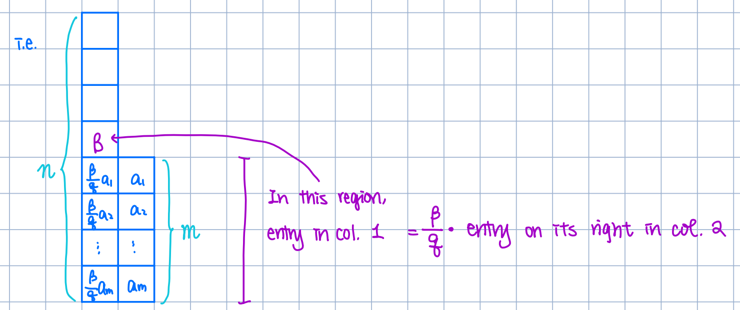
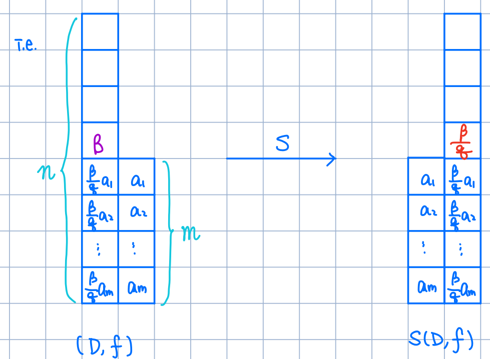
Def: For $n > m$, define $\mathcal{V}(n,m) := \{ (D,f) : D = [[n],[m]],\ f_\mu(i,1) = q^{-1} f_\mu(m+1,1)\, f_\mu(i,2)\ \ \forall\, 2 \le i \le m \}$.

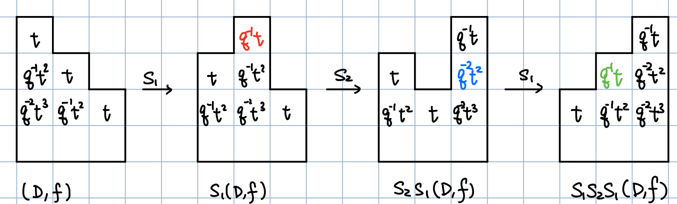


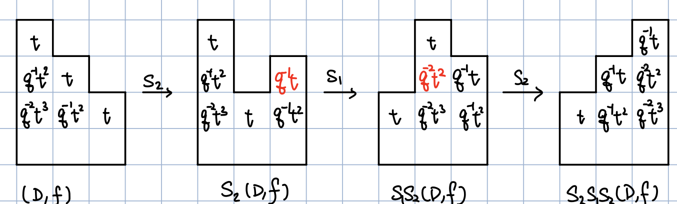
For $(D,f) \in \mathcal{V}(n,m)$, define $S(D,f)$ to be the filled diagram (D', f') obtained by interchanging column 1 and column 2 with "β" (now in column 2, i.e. cell $(m+1, 2)$) to $\frac{\beta}{q}$.



For any filled diagram (D,f), if $|D^{(j)}| > |D^{(j+1)}|$, then define $S_j(D,f)$ to be the filled diagram obtained by applying S on the subdiagram $[D^{(j)}, D^{(j+1)}]$ of D (and keep all other columns).

e.g.





Def: For $\omega \in S_n$, define $\overline{\mathrm{iDes}}(\omega) := \begin{cases} \mathrm{iDes}(\omega) \setminus \{\omega_n\} & \text{if } \omega_n = \omega_{n-1} - 1 \\ \mathrm{iDes}(\omega) & \text{otherwise} \end{cases}$ ← ignore the descent "created" by $(n\ \ n-1)$

e.g. $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$ $\therefore \overline{\mathrm{iDes}}(\omega) = \mathrm{iDes}(\omega) \setminus \{2\}$

$=3-1$

$\because \omega^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 2 & 3 \end{pmatrix}$ $\therefore \mathrm{iDes}(\omega) = \{2,3\}$ which implies $\overline{\mathrm{iDes}}(\omega) = \{2,3\} \setminus \{2\} = \{3\}$.

Since (54) creates a descent, we ignore it when counting $\overline{\mathrm{iDes}}$

Lemma: For $w \in S_n$ and $\sigma \in S_{n-2}$ s.t. $\{n-3, n-2\} = \{\sigma_{n-3}, \sigma_{n-2}\}$, let $w^{(1)} = w_{\sigma_1} \cdots w_{\sigma_{n-2}} w_{n-1} w_n$, $w^{(2)} = w_{\sigma_1} \cdots w_{\sigma_{n-2}} w_n w_{n-1}$.

If $\mathrm{iDes}(\mathrm{std}(w_1 \cdots w_{n-2})) = \mathrm{iDes}(\mathrm{std}(w_{\sigma_1} \cdots w_{\sigma_{n-2}}))$ or $\left(\text{if } \overline{\mathrm{iDes}}(\mathrm{std}(w_1 \cdots w_{n-2})) = \overline{\mathrm{iDes}}(\mathrm{std}(w_{\sigma_1} \cdots w_{\sigma_{n-2}})) \text{ and } |w_{n-2} - w_{n-3}| \neq 1\right)$, then

(a) $\mathrm{iDes}(w) = \mathrm{iDes}(w^{(1)})$

(b) $\overline{\mathrm{iDes}}(w) = \overline{\mathrm{iDes}}(w^{(2)})$

(c) $\mathrm{iDes}(w) = \mathrm{iDes}(w^{(2)})$ if $|w_n - w_{n-1}| \neq 1$

eg. $n = 9$. $w = 831792465$, $\sigma = 3145276 \Rightarrow w_{\sigma_1} \cdots w_{\sigma_7} = w_3 w_1 w_4 w_5 w_2 w_7 w_6 = 1879342 \Rightarrow \begin{cases} w^{(1)} = 187934265 \\ w^{(2)} = 187934256 \end{cases}$

$\mathrm{std}(w_1 \cdots w_7) = \mathrm{std}(8317924) = 6315724 \Rightarrow (\mathrm{std}(w_1 \cdots w_7))^{-1} = 3627415 \Rightarrow \mathrm{iDes}(\mathrm{std}(w_1 \cdots w_7)) = \{2,4,5\}$

$\mathrm{std}(w_{\sigma_1} \cdots w_{\sigma_7}) = \mathrm{std}(1879342) = 1657342 \Rightarrow (\mathrm{std}(w_{\sigma_1} \cdots w_{\sigma_7}))^{-1} = 1756324 \Rightarrow \mathrm{iDes}(\mathrm{std}(w_{\sigma_1} \cdots w_{\sigma_7})) = \{2,4,5\}$

- $w^{-1} = 362798415 \Rightarrow \mathrm{iDes}(w) = \{2,5,6,7\}$ and $\overline{\mathrm{iDes}}(w) = \{2,5,6,7\} \setminus \{5\} = \{2,6,7\}$
- $(w^{(1)})^{-1} = 175698324 \Rightarrow \mathrm{iDes}(w^{(1)}) = \{2,5,6,7\}$ and $\overline{\mathrm{iDes}}(w^{(1)}) = \{2,5,6,7\} \setminus \{5\} = \{2,6,7\}$
- $(w^{(2)})^{-1} = 175689324 \Rightarrow \mathrm{iDes}(w^{(2)}) = \{2,6,7\} = \overline{\mathrm{iDes}}(w^{(2)})$.

$\therefore \mathrm{iDes}(w) = \mathrm{iDes}(w^{(1)})$; $\overline{\mathrm{iDes}}(w) = \overline{\mathrm{iDes}}(w^{(2)})$.

Prop: For positive integers m, n s.t. $n > m$, there exists a bijection $\phi_{n,m} : S_{n+m} \to S_{n+m}$ satisfying

(ϕ1) $\mathrm{stat}_{(D,f)}(w) = \mathrm{stat}_{S(D,f)}(\phi_{n,m}(w))$ $\forall w \in S_{n+m}$ and $(D,f) \in \mathcal{V}(n,m)$ ($\phi_{n,m}$ "preserves" stat)

(ϕ2) $\mathrm{iDes}(w) = \mathrm{iDes}(\phi_{n,m}(w))$ ($\phi_{n,m}$ preserves iDes)

(ϕ3) $\{w_{n-m+2i-1}, w_{n-m+2i}\} = \{\phi_{n,m}(w)_{n-m+2i-1}, \phi_{n,m}(w)_{n-m+2i}\}$ $\forall 1 \le i \le m$ and $w_i = \phi_{n,m}(w_i)$ $\forall 1 \le i \le n-m$ ($\phi_{n,m}$ preserves the first n-m entries and "entry pairs" after the first n-m entries)

"local" condition.

By (ϕ1) and (ϕ2), we have

i.e. if D is a 2-column partition shape

$$\tilde{H}_{(D,f)}[X;q,t] = \tilde{H}_{S(D,f)}[X;q,t] \quad \text{for any } (D,f) \in \mathcal{V}(n,m)$$

By (ϕ1), (ϕ2) and (ϕ3), we have

i.e. any shape D with j^{th} col longer than $(j+1)^{th}$ column

$$\tilde{H}_{(D,f)}[X;q,t] = \tilde{H}_{S_j(D,f)}[X;q,t] \quad \text{for any filled diagram } (D,f) \text{ with } |D^{(j)}| > |D^{(j+1)}|$$

That's why (ϕ3) is "local".