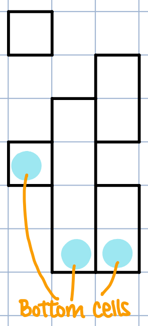
Def: • (General) diagram: a collection of cells in the 1st quadrant.

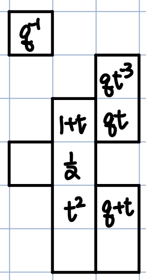
• Bottom cell : the lowest cell in a column of a diagram.

• Filled diagram: the pair (D, f) where $f : D \setminus \{\text{bottom cells of } D\} \longrightarrow \mathbb{F}$ any field (we focus on $\mathbb{C}(q,t)$)

(i.e. a diagram with a scalar in each non-bottom cell)

e.g.





A general diagram. A filled diagram

Notation: Given a diagram D, we write :

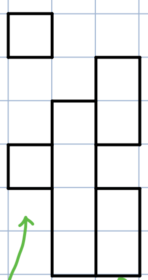
• (i,j) : cell in row i and column j (French notation)

• $D^{(j)}$: j^{th} column of D

$(D^{(j)} := \{ i : (i,j) \in D\}$ = row numbers of cells in column j of D)

e.g.

$D =$



$D^{(1)} = \{3, 6\}$

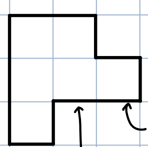
$D^{(2)} = \{1, 2, 3, 4\}$
$= [1,4] = [4]$

$D^{(3)} = \{1, 2, 4, 5\} = [1,2] \cup [4,5] = [5] \setminus \{3\}$

In this paper, we will focus on single interval columns (i.e. no "breaking" columns)

e.g.

$D =$



$D^{(3)} = [2,2]$

$D^{(2)} = [2,3]$

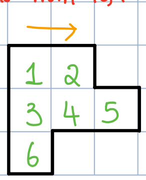
$D^{(1)} = [1,3]$

$\therefore \; D = [\,[1,3], [2,3], [2,2]\,]$

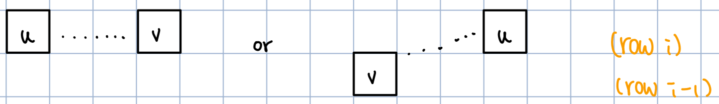
* Order the cells from left to right, starting from top row. (reading order)

e.g.

$D =$



Def: Let $\omega \in S_n$. An inversion w.r.t. ω is a pair of cells (u,v) in a diagram D (where $|D| = n$) where $\omega(u) > \omega(v)$ satisfying (Here identify a cell "u" as the "u-th" cell in reading order)

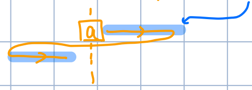


Define $\mathrm{Inv}_D(\omega) := q^{\#\text{ inversions in } D \text{ w.r.t. } \omega}$

e.g.

$D = \begin{array}{ccc} 1 & 2 & \\ 3 & 4 & 5 \\ 6 & & \end{array}$ $\quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$

$\therefore\ \omega(D) = \begin{array}{ccc} 2 & 5 & \\ 4 & 3 & 1 \\ 6 & & \end{array}$ ← For each cell $\boxed{a}$, count the no. of cells $\boxed{b}$ in



such that $b < a$.

- $\boxed{2}$: none
- $\boxed{5}$: $\boxed{4}$
- $\boxed{4}$: $\boxed{3}$ $\boxed{1}$
- $\boxed{3}$: $\boxed{1}$
- $\boxed{1}$: none
- $\boxed{6}$: none

$\therefore\ \mathrm{Inv}_D(\omega) = q^4$

Def: Let D be a diagram with n cells. For $\omega \in S_n$, a cell u in D is a decent w.r.t. ω if $\omega(u) > \omega(v)$ where v is the cell directly below u.

e.g.

$D = \begin{array}{ccc} 1 & 2 & \\ 3 & 4 & 5 \\ 6 & & \end{array}$ $\quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$

$\therefore\ \omega(D) = \begin{array}{ccc} 2 & 5 & \\ 4 & 3 & 1 \\ 6 & & \end{array}$ Cell $\boxed{2}$ is the only descent in D.

Define $\mathrm{maj}_{(D,f)}(\omega) := \prod_{\substack{u:\text{ descent in } D \\ \text{w.r.t. } \omega}} f(u)$

e.g.

$(D,f) = \begin{array}{ccc} q^2 & t & \\ q^3 t & & \\ & & \end{array}$ $\quad \omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$

$\mathrm{maj}_{(D,f)}(\omega) = t$

Define $\mathrm{stat}_{(D,f)}(\omega) := \mathrm{Inv}_D(\omega)\, \mathrm{maj}_{(D,f)}(\omega)$.

e.g.

$(D,f) = \begin{array}{ccc} q^2 & t & \\ q^3 t & & \\ & & \end{array}$

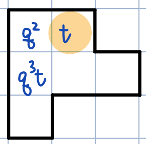
$\mathrm{stat}_{(D,f)}(\omega) = q^4 t$

Def: The (generalized) modified Macdonald polynomials $\tilde{H}_{(D,f)}(X;q,t)$ for a filled diagram (D,f) is

$$\tilde{H}_{(D,f)}[X;q,t] := \sum_{\omega \in S_{|D|}} \mathrm{stat}_{(D,f)}(\omega)\, F_{\mathrm{iDes}(\omega)}.$$

$*$ If D has a partition shape μ and $f(u) = q^{-\mathrm{arm}(u)}\, t^{\mathrm{leg}(u)+1}$, then $\tilde{H}_{(D,f^{st}_{\mu})}[X;q,t] = \tilde{H}_{\mu}[X;q,t]$.

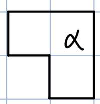
denoted f^{st}_{μ}

e.g. $(D,f) =$ 

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$$

$$\omega^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 4 & 3 & 2 & 6 \end{pmatrix} \qquad \mathrm{iDes}(\omega) = \mathrm{Des}(\omega^{-1}) = \{1,3,4\}$$

$$F_{\{1,3,4\}} = \sum_{b_1 < b_2 \le b_3 < b_4 < b_5 \le b_6} x_{b_1} x_{b_2} x_{b_3} x_{b_4} x_{b_5} x_{b_6} = x_1 x_2^2 x_3 x_4^2 + x_1 x_2 x_3 x_4 x_5^2 + \cdots$$

e.g. $(D,f) =$ 

$(\alpha \in \mathbb{F})$ $\quad (\therefore n = |D| = 3)$

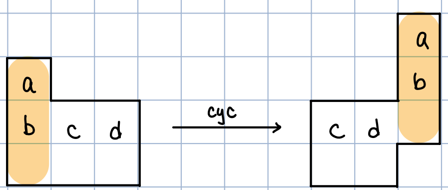
$\omega \in S_3$	id	$\begin{pmatrix}123\\132\end{pmatrix}$	$\begin{pmatrix}123\\213\end{pmatrix}$	$\begin{pmatrix}123\\231\end{pmatrix}$	$\begin{pmatrix}123\\312\end{pmatrix}$	$\begin{pmatrix}123\\321\end{pmatrix}$
$\mathrm{iDes}(\omega)$	ϕ	$\{2\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\omega(D)$	$\begin{array}{cc}1&2\\ &3\end{array}$	$\begin{array}{cc}1&3\\ &2\end{array}$	$\begin{array}{cc}2&1\\ &3\end{array}$	$\begin{array}{cc}2&3\\ &1\end{array}$	$\begin{array}{cc}3&1\\ &2\end{array}$	$\begin{array}{cc}3&2\\ &1\end{array}$
$\mathrm{Inv}_D(\omega)$	q^0	q^0	q^1	q^0	q^1	q^1
$\mathrm{maj}_D(\omega)$	1	α	1	α	1	α
$\mathrm{stat}_D(\omega)$	1	α	q	α	q	$q\alpha$

$$\therefore \tilde{H}_{(D,f)}[X;q,t] = F_{\phi} + (q+\alpha) F_{\{1\}} + (q+\alpha) F_{\{2\}} + q\alpha F_{\{1,2\}}$$
$$= h_3 + (q+\alpha) s_{21} + q\alpha e_3$$

Define $\mathrm{cyc}(D,f) := (D', f')$ where $D' = [D^{(2)}, \dots, D^{(\ell)}, D^{(1)} + 1]$ ← add 1 to the endpoints of the interval $D^{(1)}$

f': same filling as f while the last column entries in D' are the same as the 1st column in D

I.e. $\mathrm{cyc}(D,f)$ is a filled diagram obtained by moving 1st column of (D,f) to the last (rightmost) column and shift upward by 1 cell.

e.g. 

(D,f) $\qquad$ $\mathrm{cyc}(D,f)$

Lemma: Let $(D',f') = \mathrm{cyc}(D,f)$. Then for any $\omega \in S_{|D|}$, we have

because cyc preserves reading order of the cells

$$\mathrm{Inv}_D(\omega) = \mathrm{Inv}_{D'}(\omega), \quad \mathrm{maj}_{(D,f)}(\omega) = \mathrm{maj}_{(D',f')}(\omega).$$ ← because cyc preserves columns and reading order.

Hence $\tilde{H}_{(D,f)}[X;q,t] = \tilde{H}_{(D',f')}[X;q,t]$. (i.e. cyc preserves Inv and maj, and hence $\tilde{H}$)

Def: Let $w = w_1 \cdots w_n$ be a word of positive integers. A standardization $\mathrm{std}(w)$ of w is defined to be the unique $\sigma \in S_n$ s.t.

$\sigma_i < \sigma_j$ iff $w_i < w_j$ or $(w_i = w_j$ and $i < j)$

e.g. $w = 3155$ $\quad \mathrm{std}(w) = 2134$.
2134

Given $\phi : S_n \to S_n$ (view elements in S_n using one-line notation), we can extend ϕ to words with words with length n by rearranging entries of the word according to the standardization.

e.g. Given $\phi : S_4 \to S_4$ s.t. $\phi(2134) = 1423$

Then $\phi(3155) = 1535$

$\underset{2134}{} \xrightarrow{\phi} \underset{1423}{}$

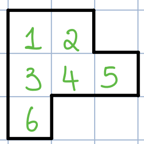
For $w \in S_n$ and $C = \{c_1 < c_2 < \cdots < c_r\} \subseteq [n]$, define $w|_C := w_{c_1} w_{c_2} \cdots w_{c_r}$

e.g. $w = 1324 \in S_4$, $C = \{2, 4\}$.

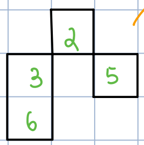
$w|_C = 34$

For a subdiagram E of a diagram D and $w \in S_{|D|}$, define $w{\downarrow}^{D}_{E} := w|_{\{N_D(u) : u \in E\}}$ $N_D(u)$ = order of cell u in reading order

e.g.

$D =$ 

$w = 254316$

$E =$ 

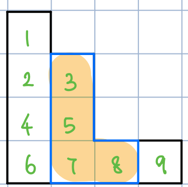
$w{\downarrow}^{D}_{E} = 5416$

For a subdiagram E of a diagram D and a map $\phi : S_{|E|} \to S_{|E|}$, define $\phi{\uparrow}^{D}_{E} : S_{|D|} \to S_{|D|}$ to be the unique map satisfying

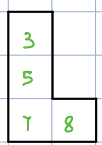
$$(\phi{\uparrow}^{D}_{E}(w)){\downarrow}^{D}_{E} = \phi(w{\downarrow}^{D}_{E}) \quad \text{and} \quad (\phi{\uparrow}^{D}_{E}(w)){\downarrow}^{D}_{D \setminus E} = w{\downarrow}^{D}_{D \setminus E}$$

(i.e. only change $w{\downarrow}^{D}_{E}$ part using extended ϕ and keep the rest)

e.g.

$D =$ 

E

$E =$ 

$$(217695348){\downarrow}^{D}_{E} = 7934$$

extended ϕ

If $\phi : S_4 \to S_4$ s.t. $3412 \mapsto 4132$, then $\phi(7934) = 9374$

$\underset{3412}{} \to \underset{4132}{}$

$$\phi{\uparrow}^{D}_{E}(217695348) = 219635748$$

only change 7934 and keep the rest