3rd day

Recall

- Column exchange rule

Prop

$$
(\mu, f_\mu) = \begin{array}[b]{|c|}\hline b_1 \\ \hline \vdots \\ \hline b_{n-m-1} \\ \hline \alpha \\ \hline a_1 \\ \hline \vdots \\ \hline a_{m-1} \\ \hline \\ \hline \end{array}\ \begin{array}[b]{|c|}\hline \alpha a_1 \\ \hline \vdots \\ \hline \alpha a_{m-1} \\ \hline \\ \hline \end{array}
\qquad
(\lambda, f_\lambda) = \begin{array}[b]{|c|}\hline \alpha a_1 \\ \hline \vdots \\ \hline \alpha a_{m-1} \\ \hline \\ \hline \end{array}\ \begin{array}[b]{|c|}\hline b_1 \\ \hline \vdots \\ \hline b_{n-m-1} \\ \hline \alpha \\ \hline a_1 \\ \hline \vdots \\ \hline a_{m-1} \\ \hline \\ \hline \end{array}
\qquad (*)
$$

Then there is a stat, iDes, content preserving bijection $\phi_{n,m} : S_{n+m} \longrightarrow S_{n+m}$, i.e. $\phi_{n,m}$ satisfies

(ϕ1) $\quad \mathrm{stat}_{(\mu, f_\mu)}(\sigma) = \mathrm{stat}_{(\lambda, f_\lambda)}(\phi_{n,m}(\sigma))$

(ϕ2) $\quad \mathrm{iDes}(\sigma) = \mathrm{iDes}(\phi_{n,m}(\sigma))$

(ϕ3) $\quad \{\sigma_i\} = \{\phi_{n,m}(\sigma)_i\}$ for $1 \le i \le m$

$\{\sigma_{n-m+2i-1}, \sigma_{n-m+2i}\} = \{\phi_{n,m}(\sigma)_{n-m+2i-1}, \phi_{n,m}(\sigma)_{n-m+2i}\}$

for $1 \le i \le m$

Rmk In particular, we have

$$\widetilde{H}_{(\mu, f_\mu)} = \widetilde{H}_{(\lambda, f_\lambda)}$$

Note that this is due to (Φ1) and (Φ2).
Since our bijection Φ preserves content as in (Φ3), this is "local", i.e., if there are two filled diagrams (D, f) and (D', f') which are identical except for two columns and those two columns are of the form $\circledast$, then we have

$$\widetilde{H}_{(D, f)} = \widetilde{H}_{(D', f')}$$

The above identity, in a special case, was studied in [HHL08] Thm 5.1.1, which states that

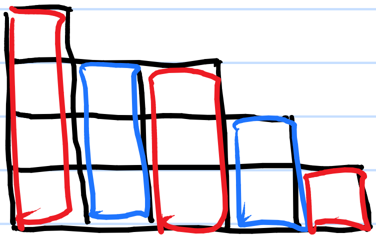
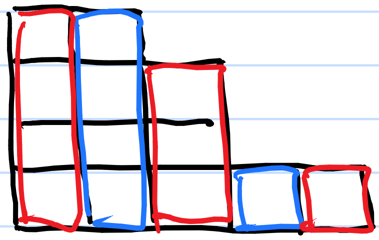
$$\widetilde{H}_{(\mu, f_\mu^{st})} = \widetilde{H}_{(\tilde{\mu}, \tilde{f})},$$

where $(\tilde{\mu}, \tilde{f})$ is obtained by applying c.e.r. to (μ, f_μ^{st})

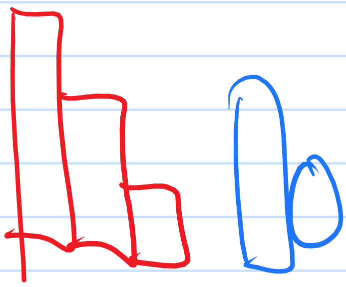
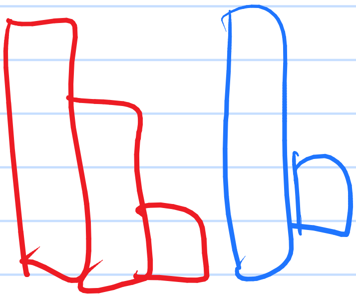
- Jim Haglund asked if there is a bijective proof for the above identity. For $t=0$ case, Alexandersson and Sawhney gave such bijection and our bijection answers Haglund's question for general t.
- Φ is used to prove other results in our paper
- condition $\circledast$ is more general

eg. $\mu = (5,4,3,1)$ $\quad f = f_{\mu}^{st}$

$\lambda = (5,3,3,2)$ $\quad g = f_{\lambda}^{st}$

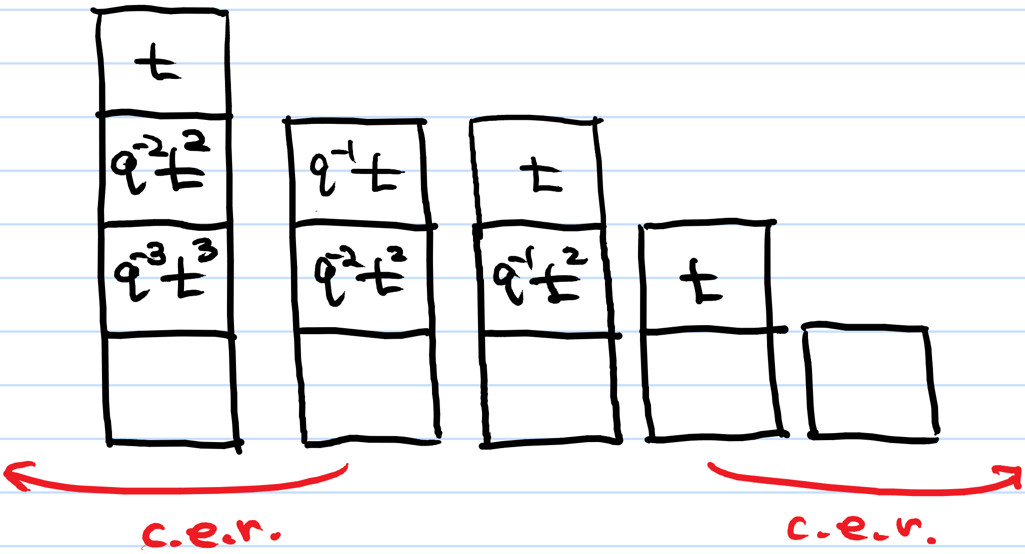
$\mu =$  $\lambda =$ 

column exchange rule

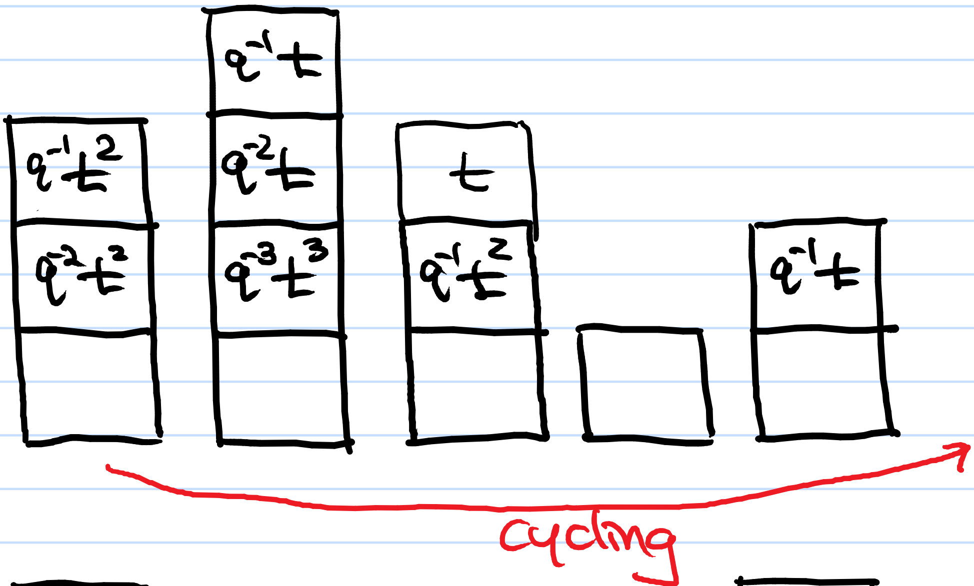
 

eg. $\mu=(5,4,3,1)$ $\quad f = f_{\mu}^{st}$

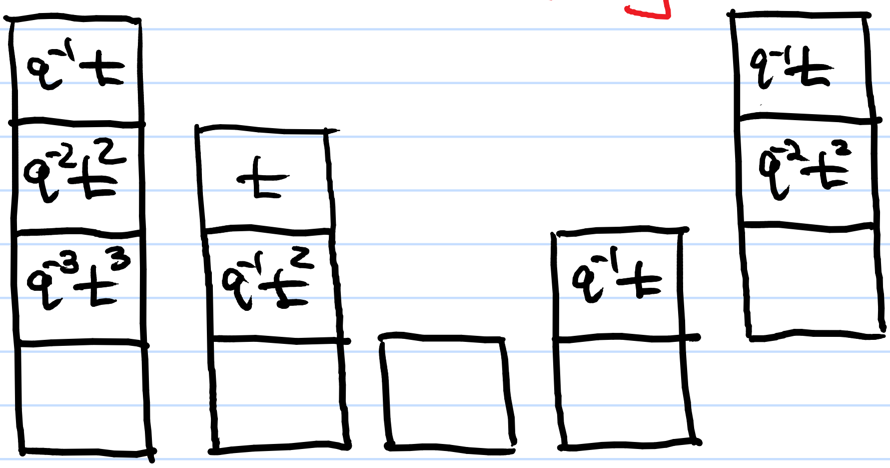
$(\mu, f) =$



$=$



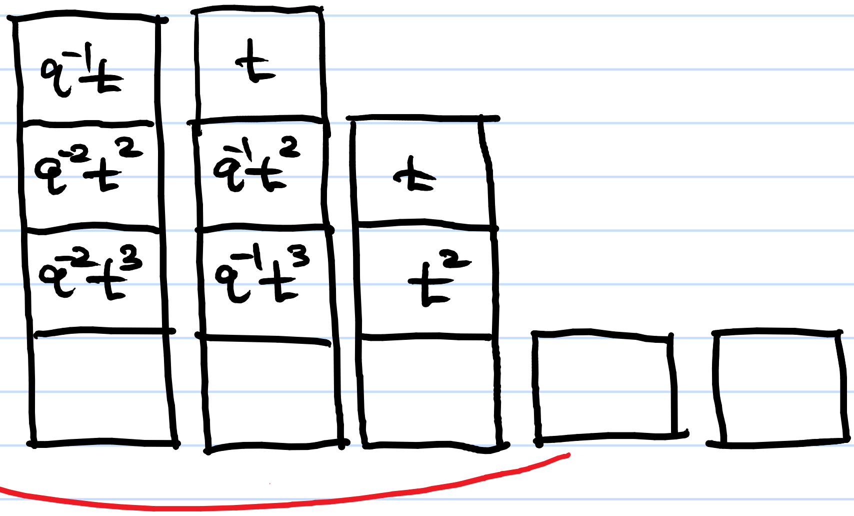
$=$



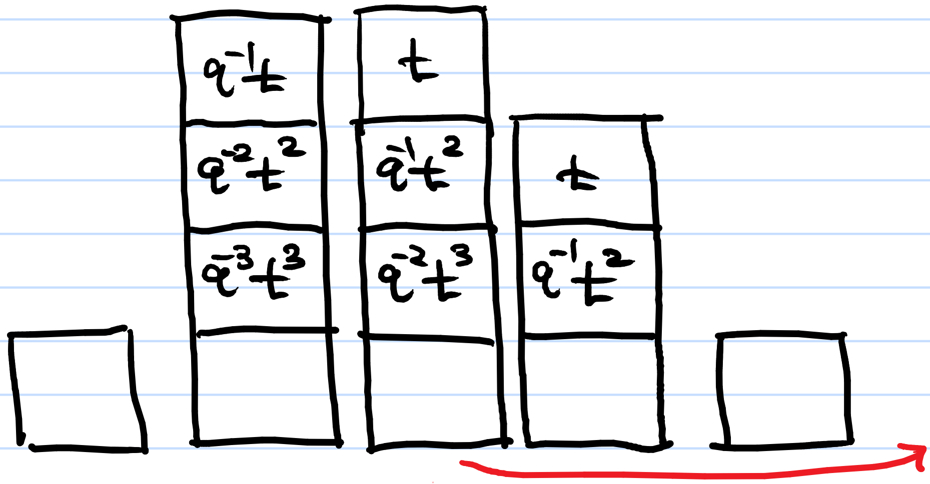
$= (\tilde{\mu}, \tilde{f})$

$\lambda = (4,3,3,2) \qquad g = f^{st}_{\lambda}$

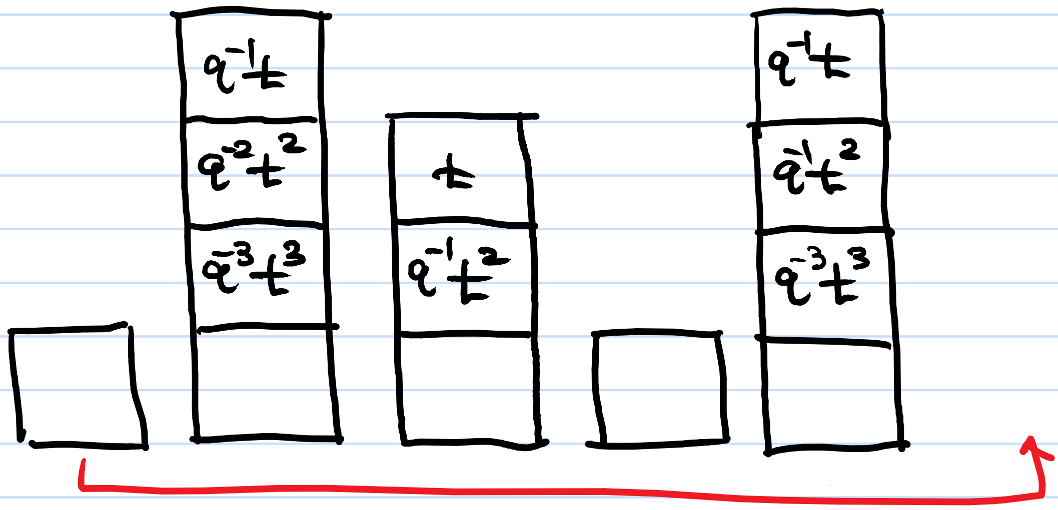
$(\lambda, g) =$



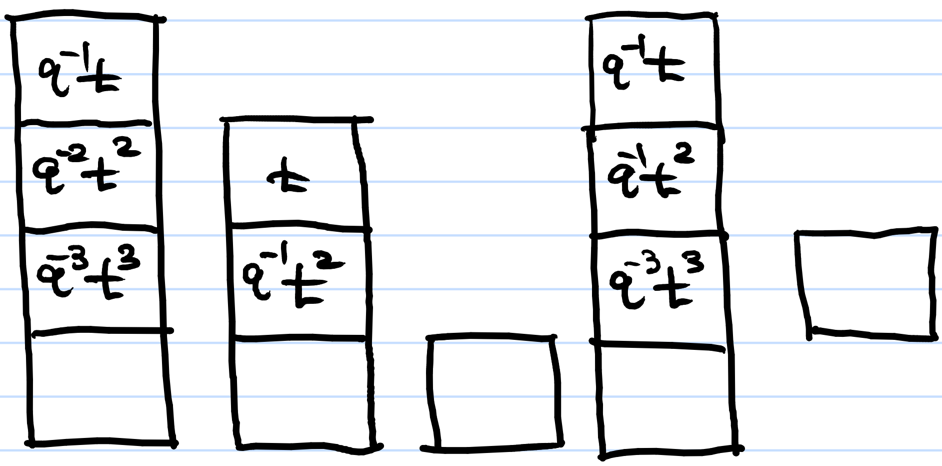
$=$



$=$



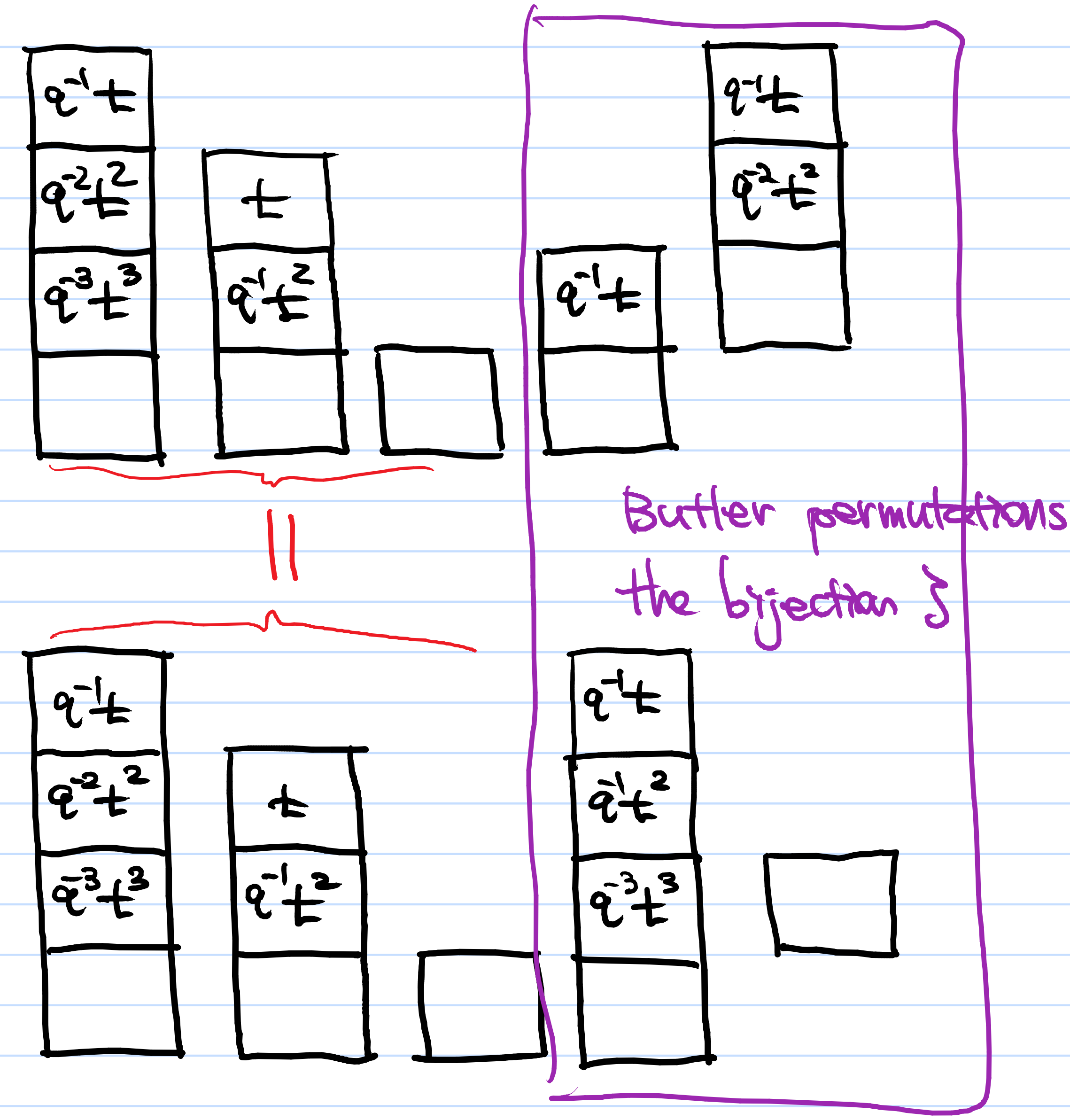
$=$



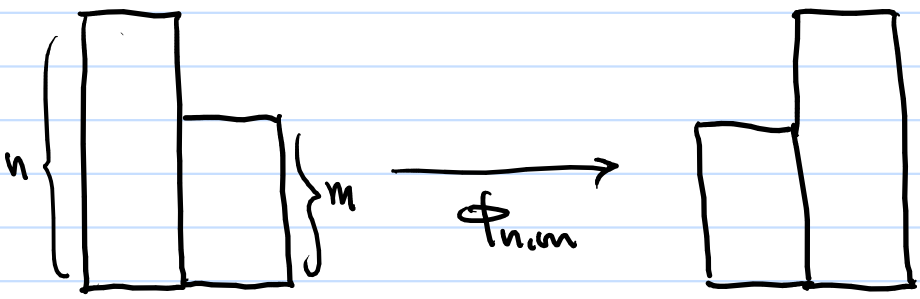
$= (\tilde{\lambda}, \tilde{g})$

$(\tilde{\mu}, \tilde{f}) =$

$(\tilde{\lambda}, \tilde{g}) =$

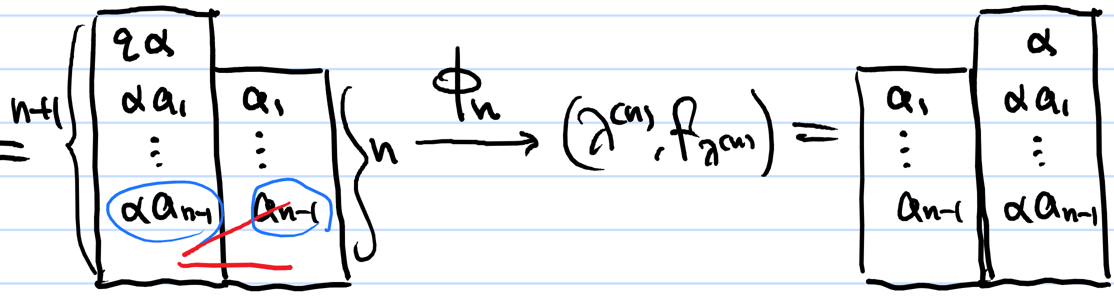


<u>Rmk</u> The condition $\circledast$ always holds while applying column exchange rule in this way [KLO22, Lemma 5.2] whose proof will be given later in this lecture series



$$\phi_{n,m}\big|_{\text{top } n-m \text{ rows}} = id$$

$$(\mu^{(n)}, f_{\mu^{(n)}}) =$$



Q. Is it possible to construct ϕ_n using ϕ_{n-1}?

$$\mathrm{stat}_{(\mu^{(n)}, f_{\mu^{(n)}})}(\omega) = \mathrm{stat}_{(\mu^{(n-1)}, f_{\mu^{(n-1)}})}\left(\omega\big|_{[2n-1]}\right)$$

$$q^{\chi(\omega_{2n-1} > \omega_{2n})}\, q^{\chi(\omega_{2n} > \omega_{2n+1})}\, (\alpha a_{n-1})^{\chi(\omega_{2n-2} > \omega_{2n})}\, a_{n-1}^{\chi(\omega_{2n-1} > \omega_{2n+1})}$$

$$= \mathrm{stat}_{(\lambda^{(n-1)}, f_{\lambda^{(n-1)}})}\left(\phi_{n-1}\left(\omega\big|_{[2n-1]}\right)\right)$$

$$q^{\chi(\omega_{2n-1} > \omega_{2n})}\, q^{\chi(\omega_{2n} > \omega_{2n+1})}\, (\alpha a_{n-1})^{\chi(\omega_{2n-2} > \omega_{2n})}\, a_{n-1}^{\chi(\omega_{2n-1} > \omega_{2n+1})}$$

$$= \mathrm{stat}_{(\lambda^{(n)}, f_{\lambda^{(n)}})}\left(\text{"}\phi_n\text{"}(\omega)\right)$$

$\mathrm{std}\left(\omega\vert_{[2n-2,2n+1]}\right)$	$\dfrac{\mathrm{stat}_{(\mu^{(n)}, f_{\mu^{(n)}})}(\omega)}{\mathrm{stat}_{(\mu^{(n-1)}, f_{\mu^{(n-1)}})}\left(\omega\vert_{[2n-3]}\right)}$	$\mathrm{std}\left(\omega\vert_{[2n-2,2n+1]}\right)$	$\dfrac{\mathrm{stat}_{(\lambda^{(n)}, f_{\lambda^{(n)}})}(\omega)}{\mathrm{stat}_{(\lambda^{(n-1)}, f_{\lambda^{(n-1)}})}\left(\omega\vert_{[2n-3]}\right)}$
1234	1	1234	1
2134	1	2134	1
1243	q	1243	q
2143	q	2143	q
1324	q	1324	q
3124	αa_n	3124	$q\alpha a_n$
1342	qa_n	1342	a_n
3142	q	3142	q
1423	qa_n	1423	$q\alpha a_n$
4123	αa_n	4123	a_n
1432	q^2a_n	1432	$q^2\alpha a_n$
4132	$q\alpha a_n$	4132	qa_n
2314	$q\alpha a_n$	2314	qa_n
3214	$q\alpha a_n$	3214	qa_n
2341	qa_n	2341	$q\alpha a_n$
3241	qa_n	3241	$q\alpha a_n$
2413	$q\alpha a_n^2$	2413	$q\alpha a_n^2$
4213	$q\alpha a_n$	4213	qa_n
2431	q^2a_n	2431	$q\alpha a_n$
4231	$q\alpha a_n^2$	4231	$q\alpha a_n^2$
3412	$q\alpha a_n^2$	3412	$q\alpha a_n^2$
4312	$q\alpha a_n^2$	4312	$q\alpha a_n^2$
3421	$q^2\alpha a_n^2$	3421	$q^2\alpha a_n^2$
4321	$q^2\alpha a_n^2$	4321	$q^2\alpha a_n^2$

$$\mathrm{stat}^{(1)}_{(D,f)}(\omega) = \begin{cases} \alpha\, \mathrm{stat}_{(D,f)}(\omega) & \text{if } \omega_{|D|-1} > \omega_{|D|} \\ q\, \mathrm{stat}_{(D,f)}(\omega) & \text{if } \omega_{|D|-1} < \omega_{|D|} \end{cases}$$

$$\mathrm{stat}^{(2)}_{(D,f)}(\omega) = \begin{cases} \mathrm{stat}_{(D,f)}(\omega) & \text{if } \omega_{|D|-1} > \omega_{|D|} \\ q\alpha\, \mathrm{stat}_{(D,f)}(\omega) & \text{if } \omega_{|D|-1} < \omega_{|D|} \end{cases}$$

$$\overline{\mathrm{iDes}}(\omega) = \begin{cases} \mathrm{iDes}(\omega) \setminus \{\omega_n\} & \text{if } \omega_{n-1} = \omega_n + 1 \\ \mathrm{iDes}(\omega) & \text{otherwise} \end{cases}$$

e.g. $\mathrm{iDes}(14532) = \{2,3\}$

$\overline{\mathrm{iDes}}(14532) = \{3\}$

<u>Claim</u> $\exists$ a bijection $\Phi_n : S_{2n+1} \longrightarrow S_{2n+1}$ satisfying

(4.1) $\mathrm{stat}^{(1)}_{(\mu^{(n)}, f_{\mu^{(n)}})}(\omega) = \mathrm{stat}^{(2)}_{(\lambda^{(n)}, f_{\lambda^{(n)}})}(\Phi_n(\omega))$

(4.2) $\overline{\mathrm{iDes}}(\omega) = \overline{\mathrm{iDes}}(\Phi_n(\omega))$

(4.3) $\{\omega_{2i}, \omega_{2i+1}\} = \{\Phi_n(\omega)_{2i}, \Phi_n(\omega)_{2i+1}\}$

$$\phi_n = \begin{cases} (\phi_{n-1}, \mathrm{Id}) & \underline{\qquad\qquad} \\ (\phi_{n-1}, S_2) & \underline{\qquad\qquad} \\ (\Psi_{n-1}, \mathrm{Id}) & \underline{\qquad\qquad} \\ (\Psi_{n-1}, S_2) & \underline{\qquad\qquad} \end{cases}$$

Case 1

$$\begin{matrix} 1234 & 1 \\ 2134 & 1 \end{matrix} \;] \longrightarrow [\; \begin{matrix} 12\boxed{34} & 1 \\ 21\boxed{34} & 1 \end{matrix}$$

$$\Phi_n(\omega) = \left(\Phi_{n-1}(\omega|_{[2n-1]}), \omega_{2n}, \omega_{2n+1}\right)$$

(Φ3) obvious

(Φ1) $$\mathrm{stat}_{(\mu(n), f_{\mu(n)})}(\omega) = \mathrm{stat}_{(\mu(n-1), f_{\mu(n-1)})}(\omega|_{[2n-1]}) \cdot 1$$

$$\overset{\text{I.H.}}{=} \mathrm{stat}_{(\lambda(n-1), f_{\lambda(n-1)})}\left(\Phi_{n-1}(\omega|_{[2n-1]})\right)$$

$$= \mathrm{stat}_{(\lambda(n), f_{\lambda(n)})}\left(\Phi_n(\omega)\right)$$

(Φ2) straightforward

Case 4

$$\begin{matrix} 3124 \\ 1342 \end{matrix} \qquad \left.\begin{matrix} \alpha a_n \\ q a_n \end{matrix}\right] \xrightarrow{\text{"}\Psi_{n-1}\text{"}} \left[\begin{matrix} 3124 \\ 1342 \end{matrix}\right. \qquad \begin{matrix} q\alpha a_n \\ a_n \end{matrix}$$

$$\Phi_n(\omega) = \begin{cases} (\Psi_{n-1}(\omega|_{[2n-1]}), m, M) & \text{if } \Psi_{n-1}(\omega|_{[2n-1]})_{2n-2} > \Psi_{n-1}(\omega|_{[2n-1]})_{2n-1} \\ (\Psi_{n-1}(\omega|_{[2n-1]}), M, m) & \text{if } \Psi_{n-1}(\omega|_{[2n-1]})_{2n-2} < \Psi_{n-1}(\omega|_{[2n-1]})_{2n-1} \end{cases}$$

$$m = \min\{\omega_{2n}, \omega_{2n+1}\}, \quad M = \max\{\omega_{2n}, \omega_{2n+1}\}$$

($\Phi 3$) obvious

($\Phi 1$)

$$\mathrm{stat}_{(\mu(n), f_{\mu(n)})}(\omega) = \begin{cases} \alpha a_n \cdot \mathrm{stat}_{(\mu(n-1), f_{\mu(n-1)})}(\omega|_{[2n-1]}) & \text{if } \omega_{2n-2} > \omega_{2n-1} \\ q a_n \cdot \mathrm{stat}_{(\mu(n-1), f_{\mu(n-1)})}(\omega|_{[2n-1]}) & \text{if } \omega_{2n-2} < \omega_{2n-1} \end{cases}$$

$$= a_n \cdot \mathrm{stat}^{(1)}_{(\mu(n-1), f_{\mu(n-1)})}(\omega|_{[2n-1]})$$

$$\overset{\text{I.H.}}{=} a_n \cdot \mathrm{stat}^{(2)}_{(\lambda(n-1), f_{\lambda(n-1)})}(\Psi_{n-1}(\omega|_{[2n-1]}))$$

$$= \begin{cases} q\alpha a_n \cdot \mathrm{stat}_{(\lambda(n-1), f_{\lambda(n-1)})}(\Phi_n(\omega)|_{[2n-1]}) \\ \qquad \text{if } \Psi_{n-1}(\omega|_{[2n-1]})_{2n-2} > \Psi_{n-1}(\omega|_{[2n-1]})_{2n-1} \\ a_n \cdot \mathrm{stat}_{(\lambda(n-1), f_{\lambda(n-1)})}(\Phi_n(\omega)|_{[2n-1]}) \\ \qquad \text{if } \Psi_{n-1}(\omega|_{[2n-1]})_{2n-2} < \Psi_{n-1}(\omega|_{[2n-1]})_{2n-1} \end{cases}$$

$$= \mathrm{stat}_{(\lambda(n), f_{\lambda(n)})}(\Phi_n(\omega))$$

$\mathrm{std}\left(\omega\vert_{[2n-2,2n+1]}\right)$	$\dfrac{\mathrm{stat}^{(1)}_{(\mu^{(n)},f_{\mu^{(n)}})}(\omega)}{\mathrm{stat}_{(\mu^{(n-1)},f_{\mu^{(n-1)}})}\left(\omega\vert_{[2n-3]}\right)}$	$\mathrm{std}\left(\omega\vert_{[2n-2,2n+1]}\right)$	$\dfrac{\mathrm{stat}^{(2)}_{(\lambda^{(n)},f_{\lambda^{(n)}})}(\omega)}{\mathrm{stat}_{(\lambda^{(n-1)},f_{\lambda^{(n-1)}})}\left(\omega\vert_{[2n-3]}\right)}$
1234	q	1234	$q\alpha$
2134	q	2134	$q\alpha$
1243	$q\alpha$	1243	q
2143	$q\alpha$	2143	q
1324	q^2	1324	$q^2\alpha$
3124	$q\alpha$	3124	$q\alpha a_n$
1342	$q\alpha a_n$	1342	$q\alpha a_n$
3142	$q\alpha a_n$	3142	q
1423	$q^2 a_n$	1423	$q^2\alpha^2 a_n$
4123	$q\alpha a_n$	4123	$q\alpha a_n$
1432	$q^2\alpha a_n$	1432	$q^2\alpha a_n$
4132	$q\alpha^2 a_n$	4132	$q a_n$
2314	$q^2\alpha a_n$	2314	$q^2\alpha a_n$
3214	$q^2\alpha a_n$	3214	$q^2\alpha a_n$
2341	$q\alpha a_n$	2341	$q\alpha a_n$
3241	$q\alpha a_n$	3241	$q\alpha a_n$
2413	$q^2\alpha a_n^2$	2413	$q^2\alpha^2 a_n^2$
4213	$q\alpha^2 a_n^2$	4213	$q^2\alpha a_n$
2431	$q^2\alpha a_n$	2431	$q^2\alpha a_n$
4231	$q^2\alpha a_n$	4231	$q\alpha a_n^2$
3412	$q^2\alpha a_n^2$	3412	$q^2\alpha a_n^2$
4312	$q^2\alpha a_n^2$	4312	$q^2\alpha^2 a_n^2$
3421	$q^2\alpha^2 a_n^2$	3421	$q^2\alpha^2 a_n^2$
4321	$q^2\alpha^2 a_n^2$	4321	$q^2\alpha a_n^2$

We can construct $\mathcal{I}_n$ using Φ_{n-1} and $\mathcal{I}_{n-1}$

Initial Case

$$(\mu^{(1)}, f_{\mu^{(1)}}) = \begin{array}{|c|c} \hline q\alpha & \\ \hline & \multicolumn{1}{c|}{} \\ \hline \end{array} \qquad (\lambda^{(1)}, f_{\lambda^{(1)}}) = \begin{array}{c|c|} \cline{2-2} & \alpha \\ \hline \multicolumn{1}{|c|}{} & \\ \hline \end{array}$$

w	$\mathrm{stat}_{(\mu^{(1)}, f_{\mu^{(1)}})}(w)$	w	$\mathrm{stat}_{(\lambda^{(1)}, f_{\lambda^{(1)}})}(w)$	$\mathrm{iDes}(w)$
123	1	123	1	ϕ
132	q	132	q	$\{2\}$
213	$q\alpha$	213	q	$\{1\}$
231	q	231	$q\alpha$	$\{1\}$
312	$q\alpha$	312	$q\alpha$	$\{2\}$
321	$q^2\alpha$	321	$q^2\alpha$	$\{1,2\}$

w	$\mathrm{stat}^{(1)}_{(\mu^{(1)}, f_{\mu^{(1)}})}(w)$	w	$\mathrm{stat}^{(2)}_{(\lambda^{(1)}, f_{\lambda^{(1)}})}(w)$	$\overline{\mathrm{iDes}}(w)$
123	q	123	$q\alpha$	ϕ
132	$q\alpha$	132	q	ϕ
213	$q^2\alpha$	213	$q^2\alpha$	$\{1\}$
231	$q\alpha$	231	$q\alpha$	$\{1\}$
312	$q^2\alpha$	312	$q^2\alpha^2$	$\{2\}$
321	$q^2\alpha^2$	321	$q^2\alpha$	$\{2\}$