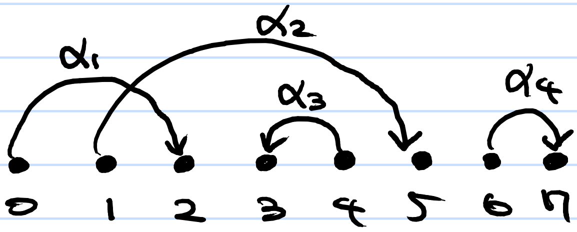
# 4th day

- **Butler permutations**

We will consider directed pefect matching.

eg.

$$\{(0,2),(1,5),(4,3),(6,7)\} =$$ 


$\alpha_1, \alpha_2, \alpha_4$ : forward direction
$\alpha_3$ : reverse direction

$\alpha_1$ and $\alpha_2$ is crossing

$\alpha_2$ and $\alpha_4$ is noncrossing

$\alpha_3$ is nested by $\alpha_2$

∘ For $w \in S_n$, consider a direct p.m.

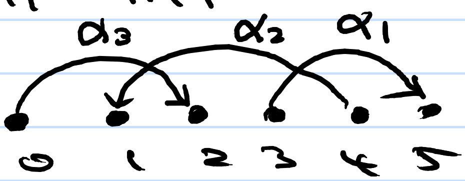
$$M(w) = \{\alpha_1(w), \cdots, \alpha_N(w)\}$$

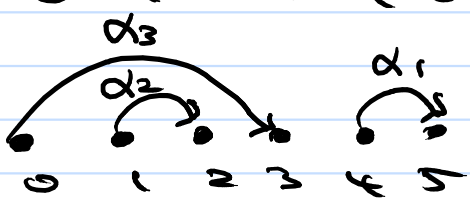
(where $N = \begin{cases} \frac{n}{2}+1 & \text{if } n: \text{even} \\ \frac{n+1}{2} & \text{if } n: \text{odd} \end{cases} = \left\lceil \frac{n+1}{2} \right\rceil$ )

on $\{0, 1, \cdots, 2N-1\}$ whose arcs are defined by

$$\alpha_{N-i}(w) = (w_{2i}, w_{2i+1}) \quad \text{for } 0 \le i \le N-1$$

where we regard $w_0 = 0$, $w_{n+1} = n+1$

eg. $M(\underline{0\ 2}\ \underline{4\ 1}\ \underline{3\ 5}) =$ 


$M(\underline{0\ 3}\ \underline{1\ 2}\ \underline{4\ 5}) =$ 

Let $k(\omega)=k$ be the smallest integer among $m$ s.t. $\alpha_m(\omega)$ and $\alpha_{m+1}(\omega)$ is noncrossing if exists.

We say $\omega$ is a <u>Butler permutation</u> if either

- $\alpha_{k+1}(\omega)$ is nested by $\alpha_k(\omega)$ and $\alpha_k(\omega)$ : reverse
- $\alpha_{k+1}(\omega)$ is not nested by $\alpha_k(\omega)$ and $\alpha_k(\omega)$ : forward

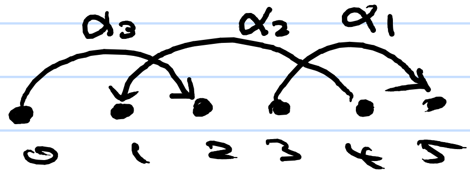
If there is no such $k$, then we say $\omega$ is a <u>Butler permutation</u> if

$$\alpha_{N-1}(\omega) \text{ is in reverse direction}$$

We denote the set of Butler permutations by $B_n$

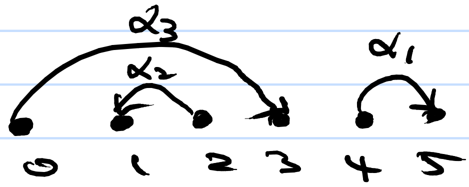
eg.

$M(0\ \underline{24}\ \underline{13}\ 5) =$



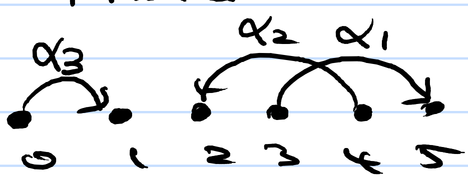

$\alpha_{N-1} = \alpha_2$ : reverse $\Longrightarrow 2413 \in B_4$

$M(\underline{0}\ \underline{32}\ \underline{14}\ 5) =$




$k=1$, $\alpha_2$ is not nested by $\alpha_1$, $\alpha_1$: forward $\Rightarrow 3214 \in B_4$

$M(\underline{0}\ \underline{14}\ \underline{23}\ 5) =$




$k=2$, $\alpha_3$ is not nested by $\alpha_2$, $\alpha_2$ = reverse $\Rightarrow 1423 \notin B_4$

<u>Observation</u> $$|B_n| = \frac{n!}{2}$$

**Prop** $\displaystyle\sum_{w\in B_n} F_{\tilde{2}Des(w)} = h_{(2,1^{n-2})}$

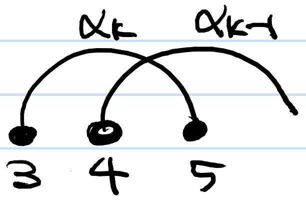
pf) $S_n' = \{w\in S_n :\ \alpha_1(w) \text{ and } \alpha_2(w) \text{ is noncrossing}\}$
$S_n'' = \{w\in S_n :\ \alpha_1(w) \text{ and } \alpha_2(w) \text{ is crossing}\}$

$B_n' = B_n \cap S_n'$, $\quad B_n'' = B_n \cap S_n''$

- $w \in B_{n+2}' \iff std(w|_{[3,n+2]}) \in B_n'$

$$\sum_{w\in B_{n+2}'} F_{\tilde{2}Des(w)} = h_{(1,1)} \cdot \sum_{w\in B_n'} F_{\tilde{2}Des(w)}$$

- flip : $B_n'' \longrightarrow S_n'' \setminus B_n''$
by reversing the direction of the arc $\alpha_k(w)$




$\Rightarrow$ We have $\tilde{2}Des(w) = \tilde{2}Des(flip(w))$

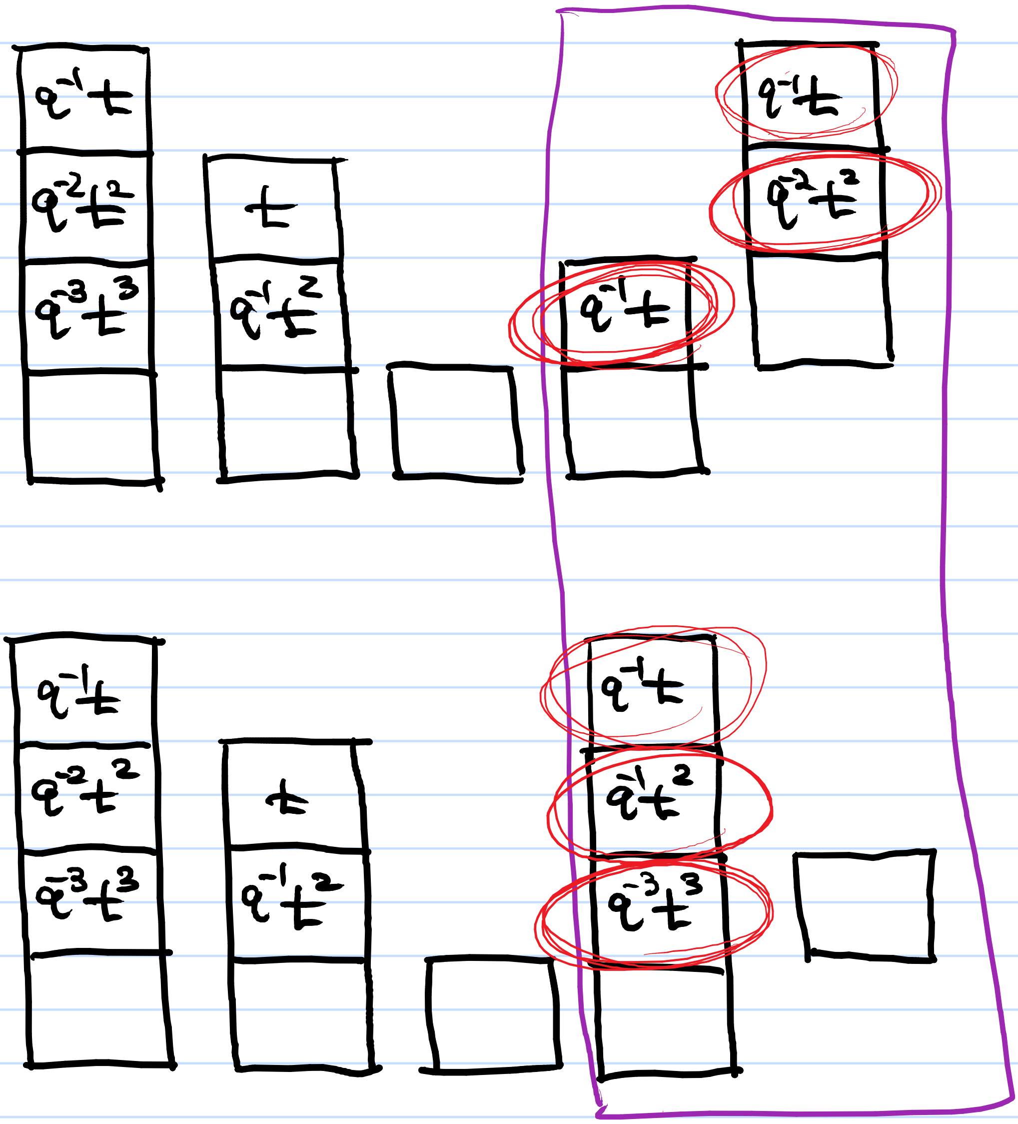
$$\sum_{w\in B_{n+2}''} F_{\tilde{2}Des(w)} = \frac{1}{2}\sum_{w\in S_{n+2}''} F_{\tilde{2}Des(w)}$$
$$= h_{(1,1)} \frac{1}{2}\sum_{w\in S_n''} F_{\tilde{2}Des(w)}$$
$$= h_{(1,1)} \sum_{w\in B_n''} F_{\tilde{2}Des(w)}$$

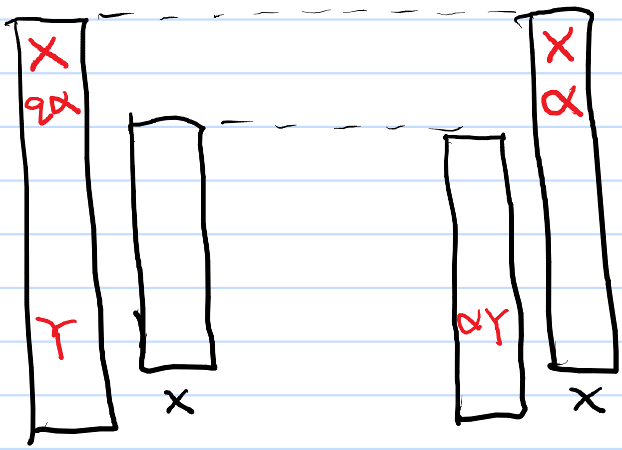
$$\sum_{w\in B_{n+2}} F_{\tilde{2}Des(w)} = h_{(1,1)} \sum_{w\in B_n} F_{\tilde{2}Des(w)}$$

(Initial conditions) $\displaystyle\sum_{w\in B_2} F_{\tilde{2}Des(w)} = h_2, \quad \sum_{w\in B_3} F_{\tilde{2}Des(w)} = h_{21}$

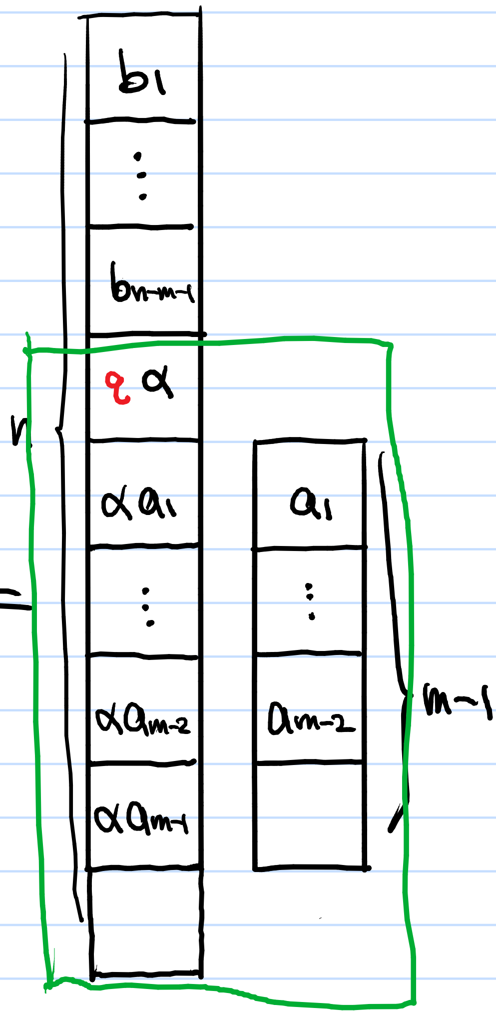
$(5,4,3,1)$ $\xrightarrow[\text{column exchange + cycling}]{}$

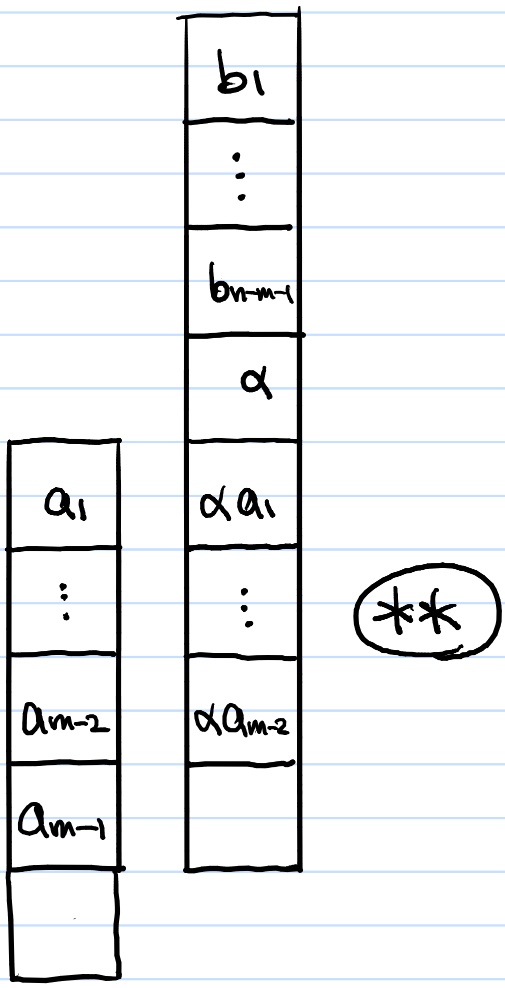
$(5,3,3,2)$ $\xrightarrow[\text{column exchange + cycling}]{}$



Prop 4.14

$(\mu, f_\mu) =$ 


$(\lambda, f_\lambda) =$  $(\ast\ast)$


There exists a bijection $\delta_{n,m} : S_{n+m-1} \longrightarrow S_{n+m-1}$ satisfying

($\delta$1) $\begin{cases} \mathrm{stat}_{(\mu,f_\mu)}(w) = \mathrm{stat}_{(\lambda,f_\lambda)}(\delta_{n,m}(w)) \text{ if } \mathrm{std}\left(w|_{[n-m,\,n+m-1]}\right) \in B_{2m} \\ \mathrm{stat}_{(\mu,f_\mu)}(w) = \alpha \cdot \mathrm{stat}_{(\lambda,f_\lambda)}(\delta_{n,m}(w)) \text{ if } \mathrm{std}\left(w|_{[n-m,\,n+m-1]}\right) \notin B_{2m} \end{cases}$

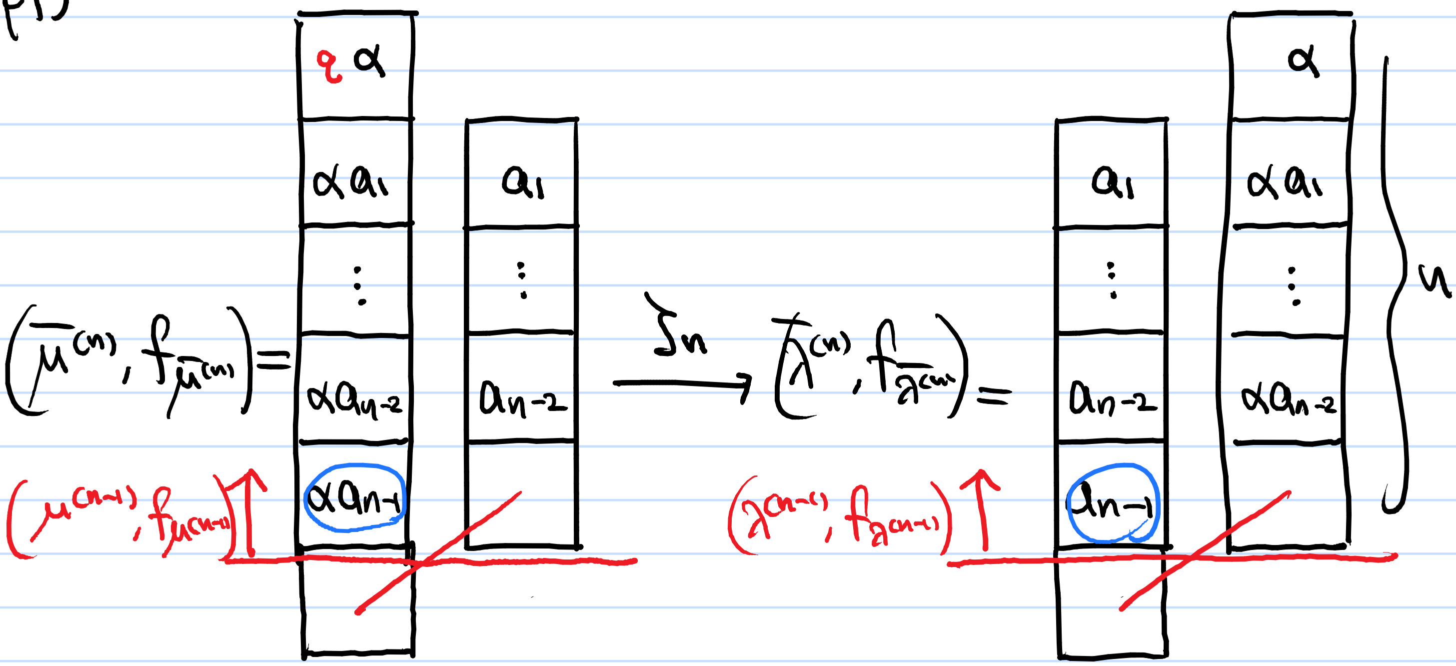
($\delta$2) $\mathrm{iDes}(w) = \mathrm{iDes}(\delta_{n,m}(w))$

($\delta$3) $\delta_{n,m}$ is content preserving

In particular, we have

$$\frac{\widetilde{H}_{(\mu,f_\mu)} - \alpha \widetilde{H}_{(\lambda,f_\lambda)}}{1-\alpha} = \sum_{\substack{w \in S_{n+m-1} \\ \mathrm{std}\left(w|_{[n-m,\,n+m-1]}\right) \in B_{2m}}} \mathrm{stat}_{(\mu,f_\mu)}(w) F_{\mathrm{iDes}(w)}$$

pf)




$$\mathrm{stat}_{(\bar{\mu}^{(n)}, f_{\bar{\mu}^{(n)}})}(\omega) = \underline{\mathrm{stat}_{(\mu^{(n-1)}, f_{\mu^{(n-1)}})}(\omega|_{[2n-1]})} \cdot q^{\chi(\omega_{2n-1} > \omega_{2n})} (\alpha a_{n-1})^{\chi(\omega_{2n-2} > \omega_{2n})}$$

$$\Big\downarrow \phi_{n-1}$$

$$\mathrm{stat}_{(\bar{\lambda}^{(n)}, f_{\bar{\lambda}^{(n)}})}(\omega) = \underline{\mathrm{stat}_{(\lambda^{(n-1)}, f_{\lambda^{(n-1)}})}(\omega|_{[2n-1]})} \cdot \underline{q^{\chi(\omega_{2n-1} > \omega_{2n})} \, a_{n-1}^{\chi(\omega_{2n-2} > \omega_{2n})}}$$

| $\mathrm{std}(\omega\|_{[2n-2,2n]})$ | | $\dfrac{\mathrm{stat}(\bar\mu^{(n)}, f_{\bar\mu^{(n)}})(\omega)}{\mathrm{stat}(\mu^{(n-1)}, f_{\mu^{(n-1)}})(\omega\|_{[2n-1]})}$ | | $\mathrm{std}(\omega\|_{[2n-2,2n]})$ | $\dfrac{\mathrm{stat}(\bar\lambda^{(n)}, f_{\bar\lambda^{(n)}})(\omega)}{\mathrm{stat}(\lambda^{(n-1)}, f_{\lambda^{(n-1)}})(\omega\|_{[2n-1]})}$ |
|---|---|---|---|---|---|
| 1 2 3 | Butler | $1$ | $\xrightarrow{\phi_{n-1}}$ | 1 2 3 | $1$ |
| 2 1 3 | | $1$ | | 2 1 3 | $1$ |
| 1<3 \| 2 | ? | $q$ | $\xrightarrow{\eta^{(1)}_{n-1}}$ | 1 3 2 | $q$ |
| 3>1 \| 2 | | $\alpha a_{n-1}$ | $\xrightarrow{\eta^{(2)}_{n-1}}$ | 3 1 2 | $a_{n-1}$ |
| 2 3 1 | not Butler | $q\alpha\, a_{n-1}$ | $\xrightarrow{\phi_{n-1}}$ | 2 3 1 | $q a_{n-1}$ |
| 3 2 1 | | $q\alpha\, a_{n-1}$ | | 3 2 1 | $q a_{n-1}$ |

**Lemma 4.15.** There exist bijections

$\eta^{(1)}: \{ w \in S_{2n+1} : w_{2n-2} < w_{2n-1} \} \longrightarrow \{ w \in S_{2n+1} : w_{2n-2} < w_{2n-1} \}$ and

$\eta^{(2)}: \{ w \in S_{2n+1} : w_{2n-2} > w_{2n-1} \} \longrightarrow \{ w \in S_{2n+1} : w_{2n-2} > w_{2n-1} \}$ s.t.

($\eta$1)
$$\begin{cases} \mathrm{stat}_{(\mu(n),\, f_{\mu(n)})}(w) = \mathrm{stat}_{(\lambda(n),\, f_{\lambda(n)})}(\eta^{(1)}_n(w)) & \text{if } w \in B_{2n+1} \\ \mathrm{stat}_{(\mu(n),\, f_{\mu(n)})}(w) = \alpha\, \mathrm{stat}_{(\lambda(n),\, f_{\lambda(n)})}(\eta^{(1)}_n(w)) & \text{if } w \notin B_{2n+1} \end{cases}$$

$$\begin{cases} \mathrm{stat}_{(\mu(n),\, f_{\mu(n)})}(w) = \mathrm{stat}_{(\lambda(n),\, f_{\lambda(n)})}(\eta^{(2)}_n(w))/\alpha & \text{if } w \in B_{2n+1} \\ \mathrm{stat}_{(\mu(n),\, f_{\mu(n)})}(w) = \mathrm{stat}_{(\lambda(n),\, f_{\lambda(n)})}(\eta^{(2)}_n(w)) & \text{if } w \notin B_{2n+1} \end{cases}$$

($\eta$2) $\quad \mathrm{iDes}(w) = \mathrm{iDes}(\eta^{(1)}_n(w))$

$\quad \mathrm{iDes}(w) = \mathrm{iDes}(\eta^{(2)}_n(w))$

($\eta$3) content preserving.

pf) We use similar arguments.

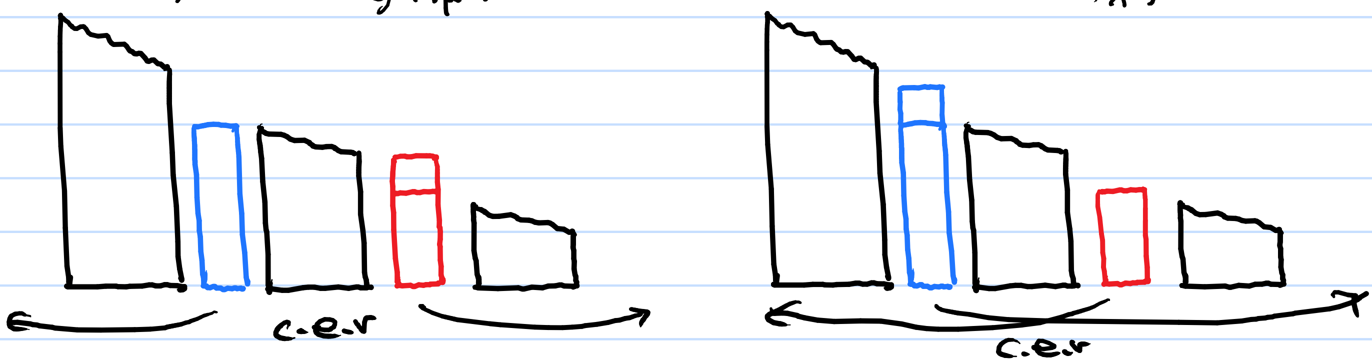
To construct $\eta^{(1)}_n$ and $\eta^{(2)}_n$ we use

$\eta^{(1)}_{n-1}$, $\eta^{(2)}_{n-1}$, $\phi_{n-1}$, and $\Phi_{n-1}$.

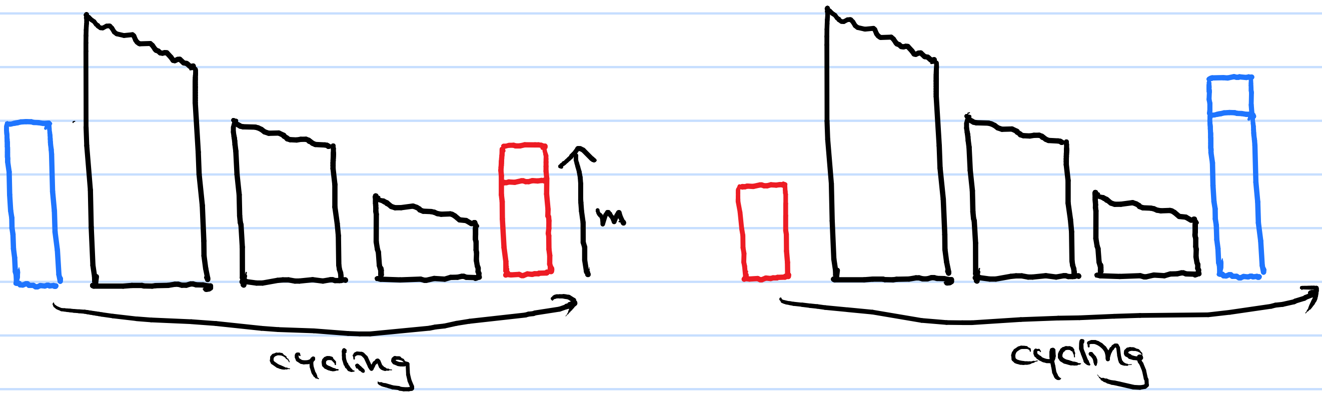
<u>pf of Prop 4.14</u>

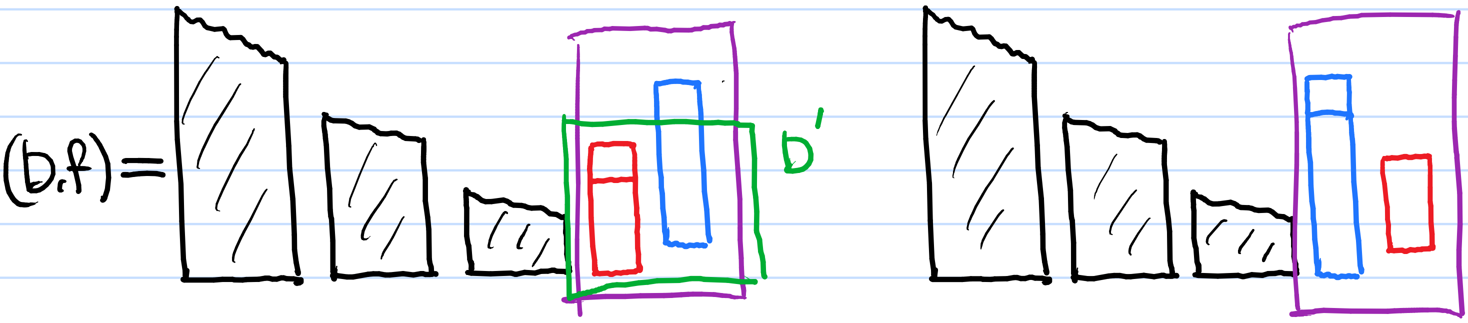
We construct $\zeta_n$ using $\eta^{(1)}_{n-1}, \eta^{(2)}_{n-1}$ and $\phi_{n-1}$

$$\tilde H_\mu = \tilde H_{(\mu, f_\mu^{st})} \qquad \tilde H_\lambda = \tilde H_{(\lambda, f_\lambda^{st})}$$




**Lemma 5.2** Condition (*) holds while applying c.e.r.




**Lemma 5.3** The filling function of shaded regions are the same. In addition, the purple regions satisfy the condition (**)

$$B_{\lambda,\mu} = \{ w \in S_n : \mathrm{std}\left( w \downarrow^{D}_{D'} \right) \in B_{2m} \}$$

Then we have

$$I_{\lambda,\mu}[X;q,t] = \frac{T_\lambda \tilde H_\mu - T_\mu \tilde H_\lambda}{T_\lambda - T_\mu} = \sum_{w \in B_{\lambda,\mu}} \mathrm{stat}_{(D,f)}(w) \, F_{i\mathrm{Des}(w)}$$

- monomial expansion

$$I_{\lambda,\mu} = \sum_{\rho \vdash n} \sum_{w \in B^{\rho}_{\lambda,\mu}} \mathrm{stat}_{\lambda,\mu}(w)\, m_\rho$$

weight $\rho$ words whose standardization is a Butler permutation of $\lambda, \mu$.

- Specialization at $q=t=1$

$$I_{\lambda,\mu}[X;1,1] = h_{(2,1^{n-2})}[X]$$

5th Lecture
- LLT polynomials and LLT equivalence
- partial results for Butler's conjecture

6th Lecture
- Combinatorial formula for $\tilde{K}_{\lambda,\mu}(q,t)$ compatible with Butler's conjecture $\mu =$ hook / 2-column