5th day

[Bat94]

$I_{\lambda,\mu}$ "not true in general" LLT s_λ

[KLO22+] [GH07]

analogous to [HHL05]

$\Downarrow$

[Hai01]

$\tilde{H}_\mu$ LLT s_λ

[HHL05] [GH07]

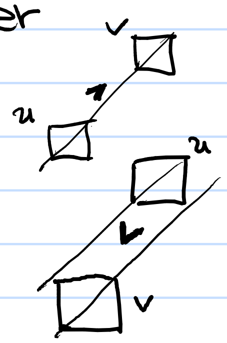
conj. [LLM03] k-Schur [BMPS]

k=2 [Lee21]

k=3 [Mil19]

- LLT polynomials.

For a tuple $\nu = (\nu^{(1)}, \nu^{(2)}, \cdots)$ of skew partitions, and its standard tableau $T = (T^{(1)}, T^{(2)}, \cdots)$, an <u>inversion</u> of T is a pair of cells $u \in \nu^{(i)}$, $v \in \nu^{(j)}$ such that $T^{(i)}(u) > T^{(j)}(v)$ and either

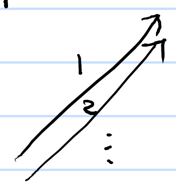


- $i < j$ and $c(u) = c(v)$ or
- $i > j$ and $c(u) = c(v) + 1$

where $c(i,j) = i - j$ is the content

$\mathrm{inv}(T) = \#$ of inversions of T

$\mathrm{rw}(T) =$ reading word of T



The <u>LLT polynomial</u> $LLT_\nu[X;q]$ is defined by

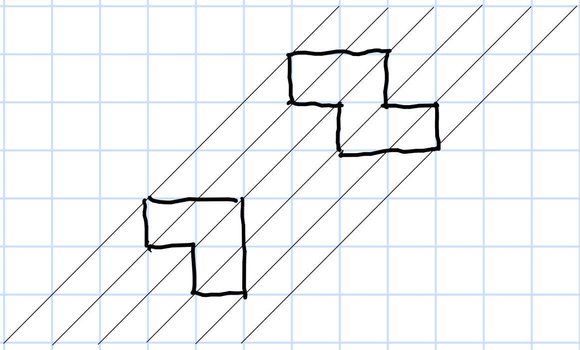
$$LLT_\nu[X;q] := \sum_{T \in SYT(\nu)} q^{\mathrm{inv}(T)} F_{i\mathrm{Des}(\mathrm{rw}(T))}$$

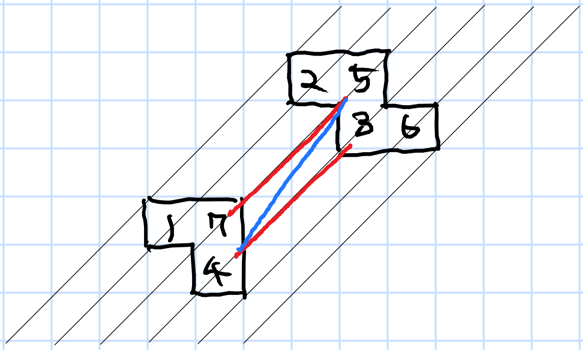
<u>Rmk</u>
- LLT polynomials are symmetric functions
- LLT polynomials are q-analogues of products of skew Schur functions:

$$LLT_\nu[X;1] = \prod_{i \geq 1} s_{\nu^{(i)}}[X]$$

- Grojnowski and Haiman proved that LLT polynomials are Schur positive [GH07]

$\nu = \left([2,2]/[1],\ [3,2]/[1] \right) =$ 

$=$ 

$T =$ 

$\mathrm{inv}(T) = 3 \qquad \mathrm{rw}(T) = 1275\underline{436} \qquad \mathrm{iDes}(\mathrm{rw}(T)) = \{3,4,6\}$

$$LLT_\nu[X;q] = \cdots + q^3 F_{\{3,4,6\}} + \cdots$$

- LLT equivalence [Mil19]

Two linear combinations

$$\sum_{\nu \in X} a_\nu(q;t)\, LLT_\nu \quad \text{and} \quad \sum_{\mu \in Y} b_\mu(q;t)\, LLT_\mu$$

are <u>LLT equivalent</u> if for every tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ of skew partitions, we have

$$\sum_{\nu \in X} a_\nu(q;t)\, LLT_{(\nu,\lambda)} = \sum_{\mu \in Y} b_\mu(q;t)\, LLT_{(\mu,\lambda)}$$

We will often write $LLT_\nu = \nu$ and LLT equivalence by

$$\sum_{\nu \in X} a_\nu(q;t)\, \nu \equiv \sum_{\mu \in Y} b_\mu(q;t)\, \mu$$

To prove an LLT equivalence, it is enough to construct a bijection $\Phi : \bigsqcup_{\nu \in X} SYT(\nu) \longrightarrow \bigsqcup_{\mu \in Y} SYT(\mu)$ satisfying

(Suppose $T \in SYT(\nu) \longmapsto \Phi(T) \in SYT(\mu)$)

(Φ1) $\quad a_\nu(q;t)\, q^{inv(T)} = b_\mu(q;t)\, q^{inv(\Phi(T))}$

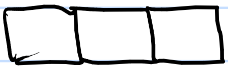
(Φ2) $\quad iDes(T) = iDes(\Phi(T))$

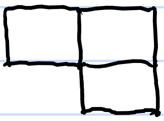
(Φ3) $\quad \{T(u) : u \in \nu,\ c(u) = m\} = \{T(u) : u \in \mu,\ c(u) = m\}$

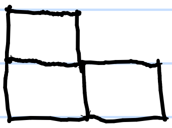
for all m.

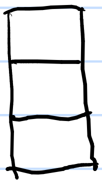
- LLT expansion of the modified Macdonald polynomials

For an interval $I = [r, s]$ and a subset $S \subseteq I \setminus \{r\}$ we define $R_I(S)$ to be the ribbon with content set I and descent set S.

eg. $R_{[3]}(\phi) =$ 

$R_{[3]}(\{2\}) =$ 

$R_{[3]}(\{3\}) =$ 

$R_{[3]}(\{2,3\}) =$ 

For a diagram D and a subset $S \subseteq D$ which contains no bottom cell, we define

$$R_D(S) = \left(R_{D^{(1)}}(S^{(1)}),\ R_{D^{(2)}}(S^{(2)}),\ \cdots \right)$$

where $D^{(i)} =$ ith column of D ← (assume this is an interval)
$S^{(i)} = S \cap D^{(i)}$

$$\tilde{H}_{(D,f)} = \sum_{\omega \in S_{|D|}} \text{inv}_D(\omega)\ \text{maj}_{(D,f)}(\omega)\ F_{i\text{Des}(\omega)}$$

$$= \sum_{\substack{S \subseteq D \\ S \text{ contains} \\ \text{no bottom cell}}} \prod_{u \in S} f(u)\ \text{LLT}_{R_D(S)}[X; q]$$

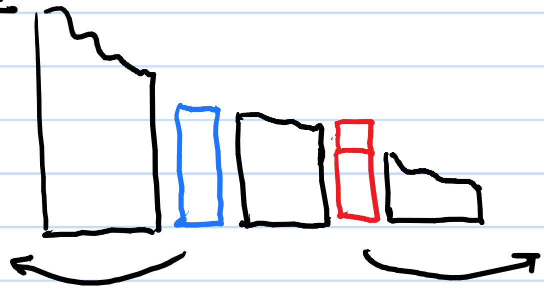
The bijection ϕ introduced in the "column exchange rule" gives an LLT equivalence between

$\tilde{H}_{(\mu, f_\mu)}$ and $\tilde{H}_{(\lambda, f_\lambda)}$

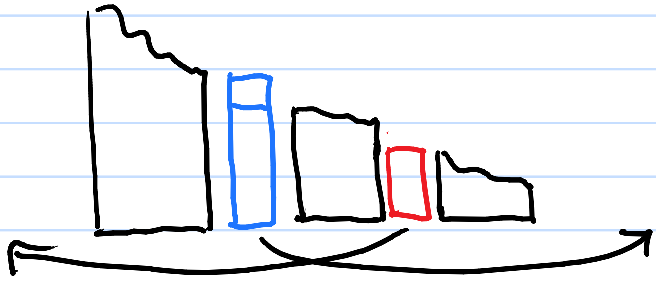
where (μ, f_μ) and (λ, f_λ) satisfy ⊛

- Partial results for Butler's conjecture

Recall

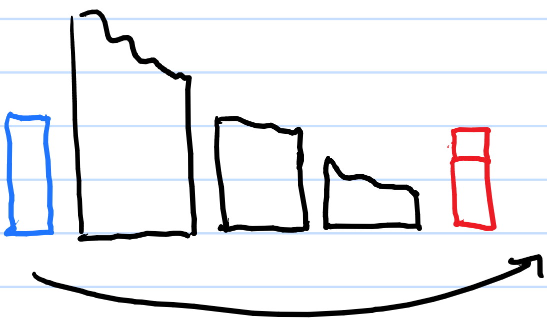


column exchange

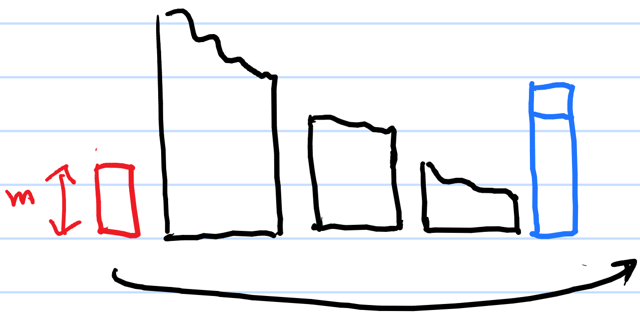


column exchange

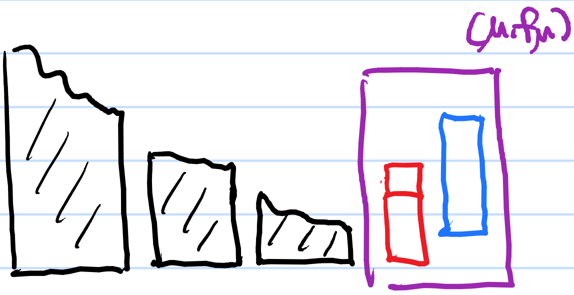
Condition ⊛ holds while applying column exchange rule (Lemma 5.2)

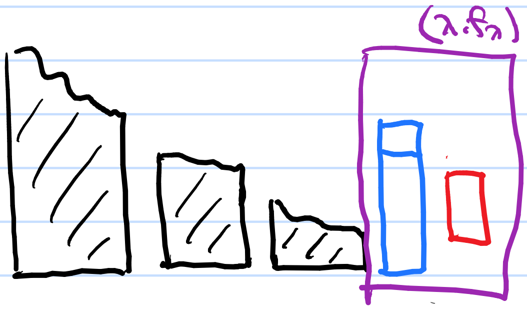


cycling



cycling





satisfy (**)

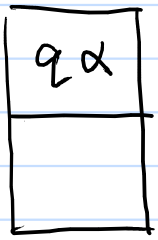
Suppose we can prove:
For (D,f) and (D',f') satisfying (**),

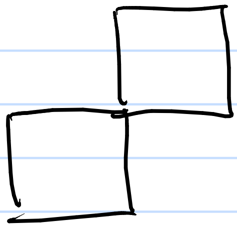
$$\frac{\tilde{H}_{(D,f)} - \alpha \cdot \tilde{H}_{(D',f')}}{1-\alpha}$$

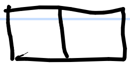
is LLT equivalent to a positive linear combination of LLT polynomials.
Then we have s-positivity of $I_{\lambda,\mu}$

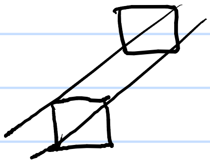
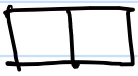
We prove this claim for $m=1$ / $m=2$

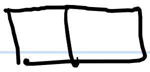
proof) $m=1$.

$(D,f) =$ 

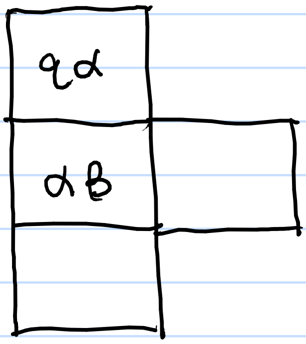
$(D',f') =$ 

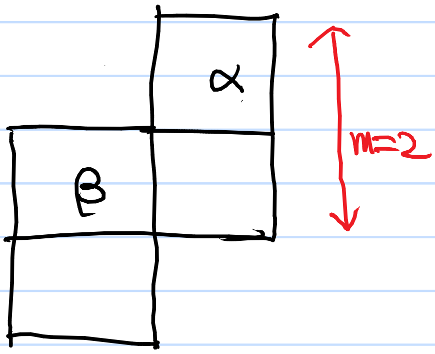
$\widetilde{H}_{(D,f)} = q\alpha$  $+$ 

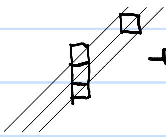
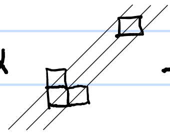
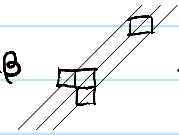
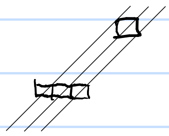
$\widetilde{H}_{(D',f')} =$  $\equiv q$  $+$ 

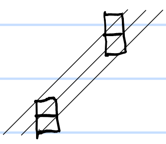
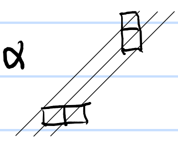
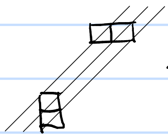
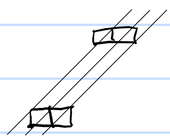
$$\frac{\widetilde{H}_{(D,f)} - \alpha \widetilde{H}_{(D',f')}}{1-\alpha} =$$ 

$m=2$

$(D,f) =$ 

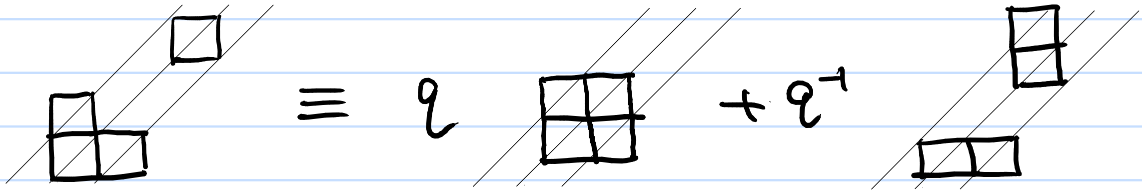
$(D',f') =$ 

$\widetilde{H}_{(D,f)} = q\alpha^2\beta$  $+ q\alpha$  $+ \alpha\beta$  $+$ 

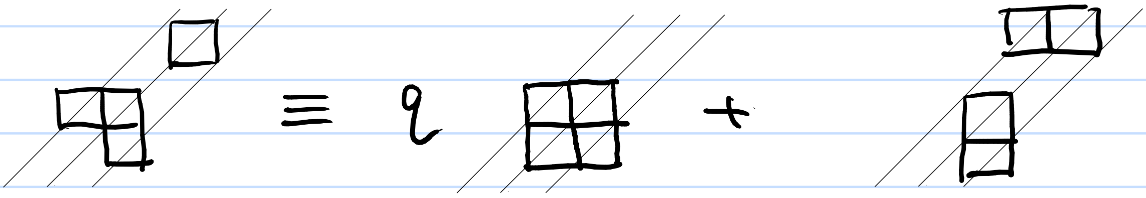
$\widetilde{H}_{(D',f')} = \alpha\beta$  $+ \alpha$  $+ \beta$  $+$ 

Prop 6.3. [Miller 19]

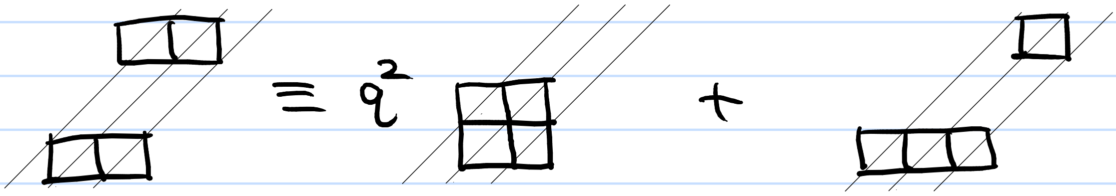
(1)



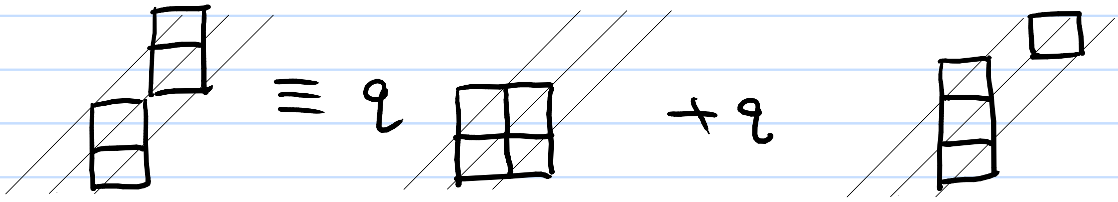
(2)



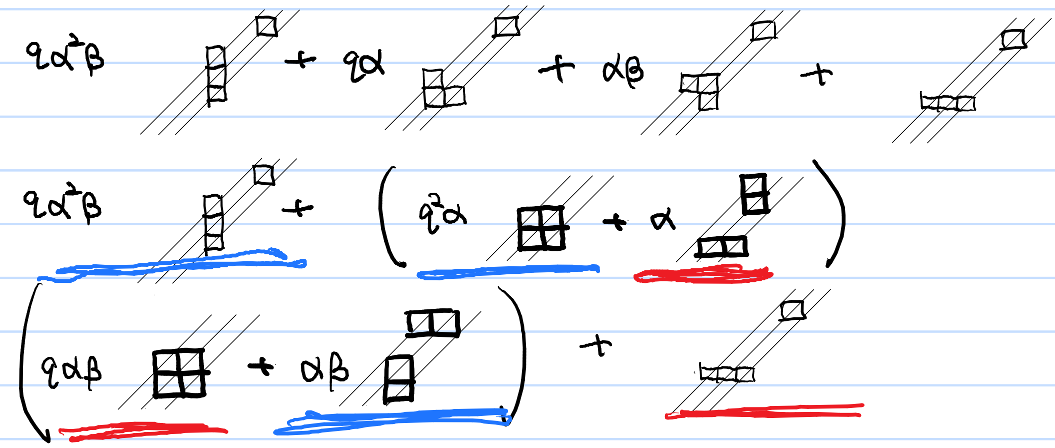
(3)



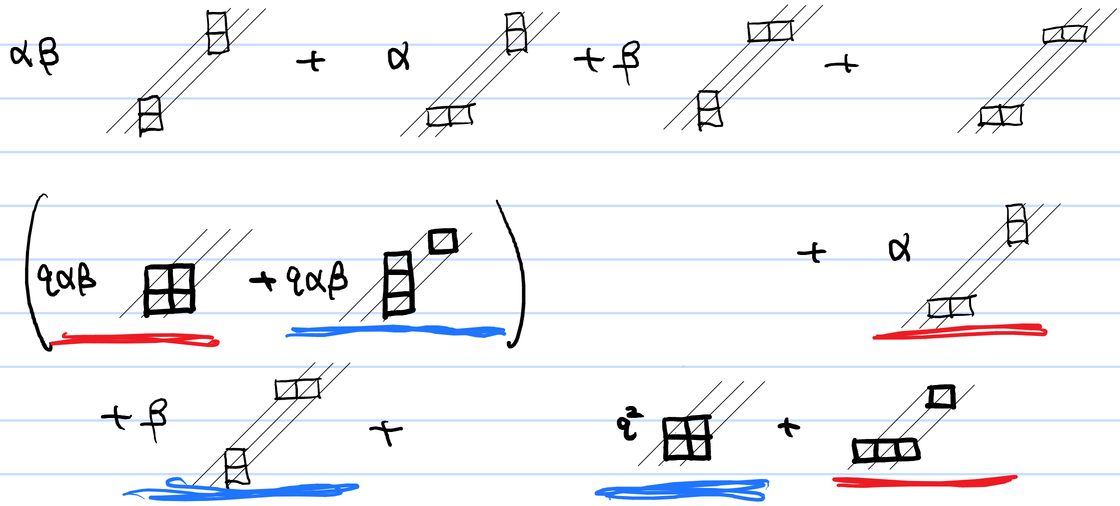
(4)



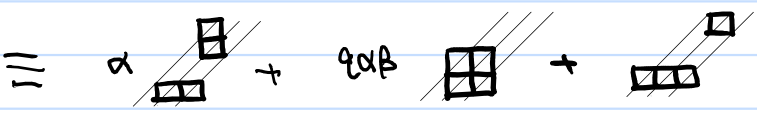
$\tilde{H}_{(D,f)} =$



$\tilde{H}_{(D',f')} =$



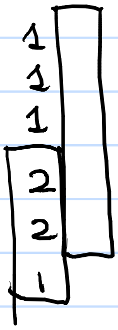
$\dfrac{\tilde{H}_{(D,f)} - \alpha \tilde{H}_{(D',f')}}{1-\alpha} \equiv$



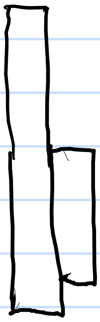
Rmk

$$[t^5]\left(\frac{\tilde H_{(2,2,2)} - \frac{q}{t}\tilde H_{(2,2,1,1)}}{1-\frac{q}{t}}\right) = s_{321} + q\, s_{2211}$$

This cannot be written as a positive linear combination of LLT polynomials indexed by tuples of content (1,2,2,1)



and



Cor $I_{\lambda,\mu}[X;q,1]$ is Schur positive

pf) $\tilde H_\mu[X;q,1] = \prod_{i=1}^{\ell(\mu)} \tilde H_{(\mu_i)}[X;q,1]$

$$I_{\lambda,\mu}[X;q,1] = \underline{\prod_{k\neq i,j}\tilde H_{(\mu_k)}[X;q,1]}\ \underline{I_{(\lambda_i,\lambda_j),(\mu_i,\mu_j)}[X;q,1]}$$

$I_{(\lambda_i,\lambda_j),(\mu_i,\mu_j)}[X;q,t]$ is s-positive from the previous claim, so this completes the proof.

Using (q,t)-symmetry : $\tilde H_\mu[X;q,t] = \tilde H_{\mu'}[X;t,q]$

Cor $I_{\lambda,\mu}[X;1,t]$ is Schur positive.