Last day

For a partition $\mu \vdash n$ let

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q,t)\, s_\lambda .$$

__Problem__ Find a combinatorial interpretation for $\tilde{K}_{\lambda\mu}(q,t)$

Butler's conjecture suggests that there might be a way to anticipate "new" $\tilde{K}_{\lambda\mu^{(2)}}$ from "known" $\tilde{K}_{\lambda\mu^{(1)}}(q,t)$

eg. Table of $\tilde{K}_{\lambda\mu}(q,t)$ for $\mu^{(1)} = (2,2)$, $\mu^{(2)} = (3,1)$

$\lambda \backslash \mu$	$\begin{array}{\|c\|c\|} \hline t & qt \\ \hline 1 & q \\ \hline \end{array}$		$\begin{array}{\|c\|c\|c\|} \hline t & & \\ \cline{1-1} 1 & q & q^2 \\ \hline \end{array}$
(4)	1	$\xrightarrow{=}$	1
$(3,1)$	qt	$\xrightarrow{q/t}$	q^2
	q	$\xrightarrow{=}$	q
	t	$\xrightarrow{=}$	t
$(2,2)$	q^2	$\xrightarrow{=}$	q^2
	t^2	$\xrightarrow{q/t}$	qt
$(2,1,1)$	q^2t	$\xrightarrow{q/t}$	q^3
	qt^2	$\xrightarrow{q/t}$	q^2t
	qt	$\xrightarrow{=}$	qt
$(1,1,1,1)$	q^2t^2	$\xrightarrow{q/t}$	q^3t

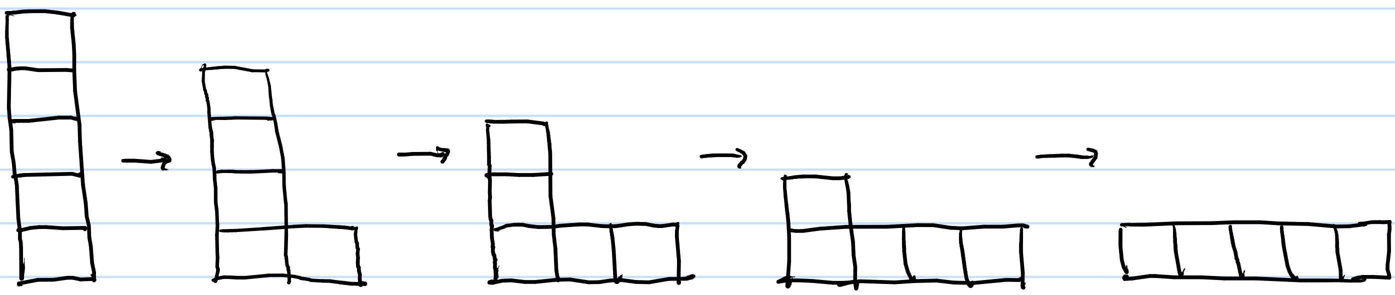
Mathematician!

L. Butler and L. Butler

For instance we know that

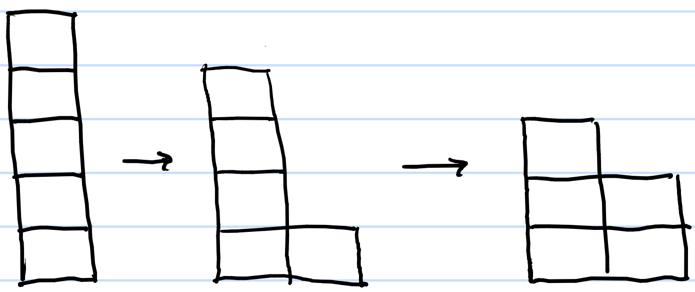
$$\tilde{K}_{\lambda,(1^n)}(q,t) = \sum_{T \in SYT(\lambda)} t^{maj(T)}$$

From (1^n), by moving a cell one by one we obtain "hooks"



Sami Assaf in her 2018 paper gave a combinatorial formula for $\tilde{K}_{\lambda\mu}(q,t)$ μ=hook. In [Sec 7.2, KLO22+] we showed that her formula is compatible w/ Butler's conjecture.

From (1^n), by moving cells one by one, we obtain "two-columns"



- k-Schur functions

$\mathrm{Par}^k := \{ \text{partitions w/ its parts} \le k \}$

For a $\mu \in \mathrm{Par}^k$, there is a family of symmetric functions $s_\mu^{(k)}[X;t]$ called the k-Schur functions

<u>Conj</u> [Lapointe – Lascoux – Morse 03] For $\mu \in \mathrm{Par}^k$,

$\omega \tilde{H}_\mu$ is k-Schur positive.

<u>Thm</u> [Blasiak – Morse – Pun – Summers 19]

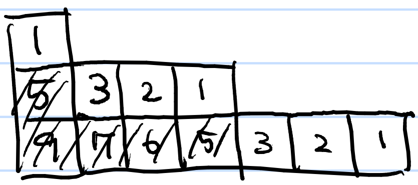
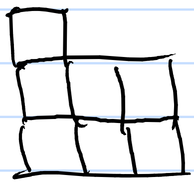
- $s_\mu^{(k)}$ is Schur positive
- (shift invariance) For $\ell(\mu) \le \ell$

$$e_\ell^\perp \, s^{(k+1)}_{\mu + (1^\ell)} = s^{(k)}_\mu$$

- (Vertical dual Pieri rule)

$$e_d^\perp \, s_\mu^{(k)} = \sum_{T \in \boxed{VSMT^k_{(d)}(\mu)}} t^{\boxed{\mathrm{spin}(T)}} \, s^{(k)}_{\boxed{\mathrm{inside}(T)}}$$

<u>Recall</u> $\mathfrak{p} : \{ (k+1)\text{-cores} \} \xrightarrow{\text{bij}} \mathrm{Par}^k$

(3+1)-core  $\longmapsto$  3-bounded

- A <u>strong cover</u> $\tau \Rightarrow \kappa$ is a pair of $(k+1)$-cores s.t.

$\tau \subseteq \kappa$ and $|\mathfrak{p}(\tau)| + 1 = |\mathfrak{p}(\kappa)|$

- A <u>strong marked cover</u> $\tau \overset{r}{\Rightarrow} \kappa$, $r \in \left\{ \begin{array}{l} \text{smallest row index of} \\ \text{connected comp. of } \kappa/\tau \end{array} \right\}$
- A <u>vertical strong marked tableau</u> (VSMT) of weight η is

$$T = (\kappa^{(0)} \overset{r_1}{\Rightarrow} \kappa^{(1)} \overset{r_2}{\Rightarrow} \cdots \overset{r_m}{\Rightarrow} \kappa^{(m)})$$

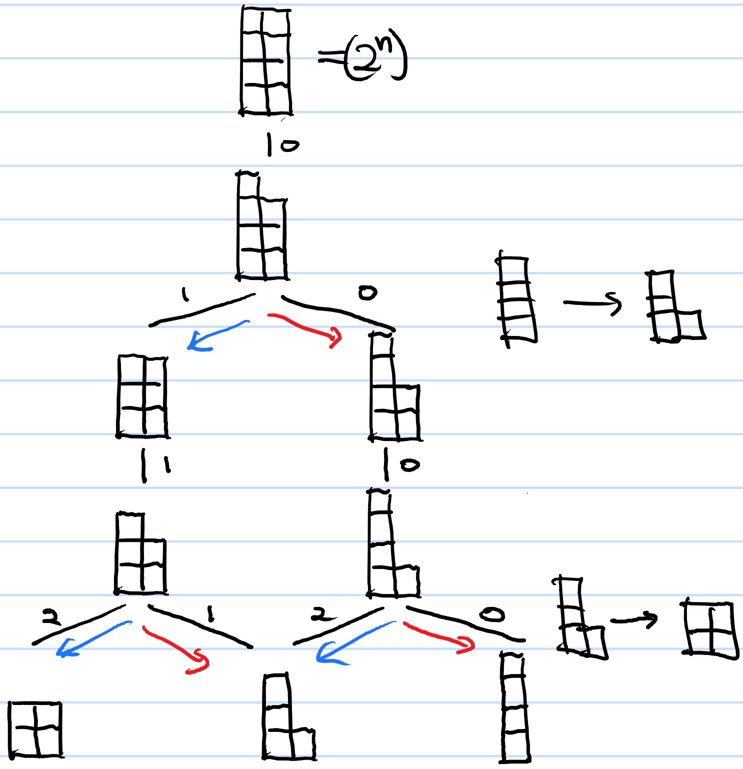
such that $r_1 < r_2 < \cdots < r_{\eta_1}$, $r_{\eta_1+1} < r_{\eta_1+2} < \cdots < r_{\eta_1+\eta_2}$, $\cdots$

- $VSMT^k_\eta(\mu) = \{ \text{VSMT } T\text{'s of weight } \eta \text{ and } \mathfrak{p}(\kappa^{(m)}) = \mu \}$

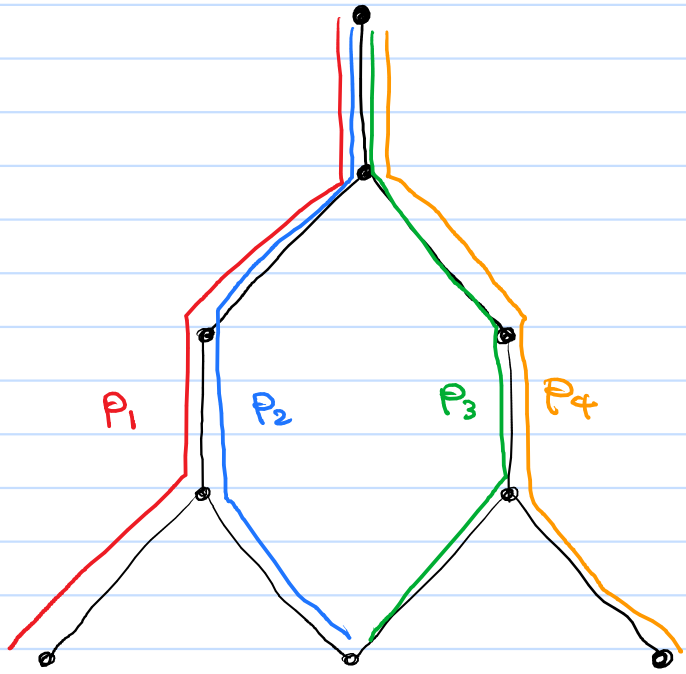
$\mathrm{inside}(T) = \mathfrak{p}(\kappa^{(0)})$

- $\mathrm{spin}(\tau \overset{r}{\Rightarrow} \kappa) = c \cdot (h-1) + N$, $\mathrm{spin}(T) := \sum \mathrm{spin}(\kappa^{(i)} \overset{r_{i+1}}{\Rightarrow} \kappa^{(i+1)})$

$$\omega \tilde{H}_{(1^n)} = S^{(1)}_{(1^n)} \underset{\text{shift inv.}}{=} e_n^{\perp} S^{(2)}_{(2^n)} = \sum_{\text{V.d.Rmle } T \in VSMT^{2}_{(n)}(2^n)} t^{\text{spin}(T)} S^{(2)}_{\text{inside}(T)}$$



1 path 2 paths 1 path



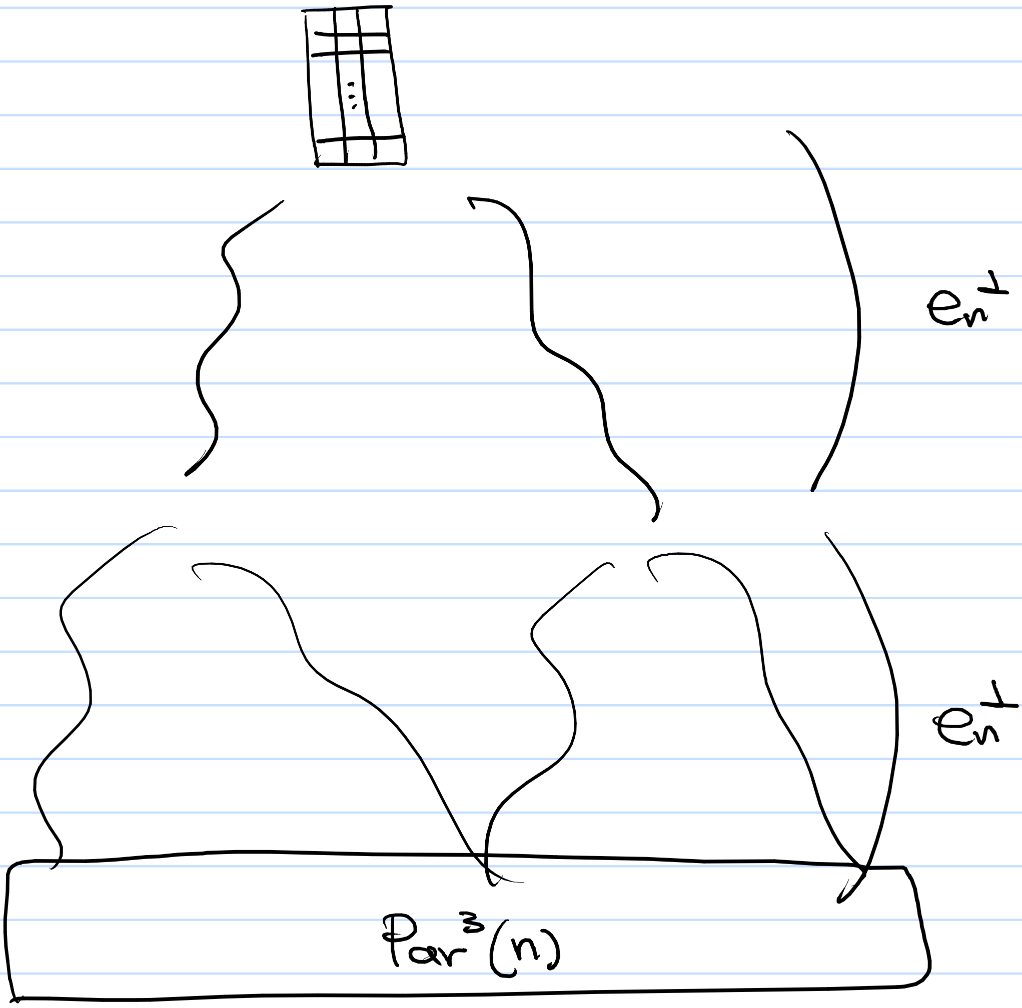
$$\omega \tilde{H}_{(1^4)} = \sum_{P:\text{path}} t^{\text{spin}(P)} S^{(2)}_{\text{end}(P)}$$

$$= t^4 S^{(2)}_{22} + (t^3+t^2) S^{(2)}_{211} + 1 \cdot S^{(2)}_{1111}$$

$\lambda \in Par^2$	Paths	$\omega\tilde{H}_{1111}$		$\omega\tilde{H}_{211}$		$\omega\tilde{H}_{22}$
22	P_1	t^4	$\xrightarrow{q/t^3}$	qt	$\xrightarrow{q/t}$	q^2
211	P_2	t^3	$\xrightarrow{q/t^3}$	q	$\xrightarrow{=}$	q
	P_3	t^2	$\xrightarrow{=}$	t^2	$\xrightarrow{q/t}$	qt
1111	P_4	1	$\xrightarrow{=}$	1	$\xrightarrow{=}$	1

This formula is equiv. to Zabrocki's formula

$$\omega\tilde{H}_{(1^n)} = s^{(1)}_{(1^n)} = (e_n^\perp)^2\, s^{(3)}_{(3^n)}$$



$$\omega\tilde{H}_{(1^n)} = \sum_{P:\text{path}} t^{\mathrm{spin}(P)}\, s^{(3)}_{\mathrm{end}(P)}$$