Theorem (Littlewood 1958, Scharf/Thibon 1994): For $f \in \Lambda$ and $g \in \Lambda_n$ for some $n \in \mathbb{N}$, $\langle \phi_n(f), g \rangle = \langle f, g[1+h_1+h_2+\cdots] \rangle$.

Proof: It suffices to consider the case when $f = p_\lambda$ and $g = p_\mu$ for some $\mu \vdash n$ (Result then follows from linearity of ϕ_n and bi-linearity of $\langle \cdot , \cdot \rangle$)

Let $m_i = \#\, i$ in μ. $\quad (\therefore \mu = (\underbrace{n, \ldots, n}_{m_n}, \underbrace{n-1, n-1, \ldots, n-1}_{m_{n-1}}, \ldots, \underbrace{2, 2, \ldots, 2}_{m_2}, \underbrace{1, 1, \ldots, 1}_{m_1}))$

Then
$$p_\mu[1+h_1+h_2+\cdots] = p_\mu[(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots]$$
$$= p_\mu[\exp(\ln(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)\cdots)]$$
$$= p_\mu\left[\exp\left(\ln\frac{1}{1-x_1} + \ln\frac{1}{1-x_2} + \cdots\right)\right]$$
$$= p_\mu\left[\exp\left(x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \cdots + x_2 + \frac{x_2^2}{2} + \frac{x_2^3}{2} + \cdots + \cdots\right)\right]$$
$$= p_\mu\left[\exp\left(x_1 + x_2 + \cdots + \frac{x_1^2}{2} + \frac{x_2^2}{2} + \cdots + \frac{x_1^3}{3} + \frac{x_2^3}{3} + \cdots + \cdots\right)\right]$$
$$= p_\mu\left[\exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right)\right]$$
$$= \left(p_1\left[\exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right)\right]\right)^{m_1}\left(p_2\left[\exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right)\right]\right)^{m_2}\cdots\left(p_n\left[\exp\left(p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{3}p_3 + \cdots\right)\right]\right)^{m_n}$$
$$= \prod_{i=1}^{n} p_i\left[\exp\left(\sum_{j=1}^{\infty}\frac{p_j}{j}\right)\right]^{m_i}$$
$$= \prod_{i=1}^{n} p_i\left[\prod_{j=1}^{\infty}\exp\left(\frac{p_j}{j}\right)\right]^{m_i}$$

$p_i[AB] = p_i[A]\,p_i[B]$

$$= \prod_{i=1}^{n}\prod_{j=1}^{\infty} p_i\left[\exp\left(\frac{p_j}{j}\right)\right]^{m_i}$$

$$p_i\left[\exp\left(\frac{p_j}{j}\right)\right] = p_i\left[1 + \frac{p_j}{j} + \frac{p_j^2}{2!\,j^2} + \cdots\right]$$
$$= 1 + \frac{1}{j}p_j(x_1^i, x_2^i, \ldots) + \frac{1}{2!\,j^2}p_j^2(x_1^i, x_2^i, \ldots) + \cdots$$
$$= 1 + \frac{1}{j}(x_1^{ij} + x_2^{ij} + \cdots) + \frac{1}{2!\,j^2}\cdot(x_1^{ij} + x_2^{ij} + \cdots)^2 + \cdots$$
$$= \exp\left(\frac{x_1^{ij} + x_2^{ij} + \cdots}{j}\right) = \exp\left(\frac{p_{ij}}{j}\right)$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{\infty}\left(\exp\frac{p_{ij}}{j}\right)^{m_i} = \prod_{i,j\geq 1}\exp\left(\frac{m_i p_{ij}}{j}\right) = \prod_{k\geq 1}\prod_{d\mid k}\exp\left(\frac{m_d p_k}{(k/d)}\right) = \prod_{k\geq 1}\prod_{d\mid k}\exp\left(d m_d \cdot \frac{p_k}{k}\right)$$

b/c for $i > n$, $m_i = 0$

let $ij = k$, $i = d$
$\therefore j = \frac{k}{d}$

$$\therefore \langle p_\lambda, p_\mu[1+h_1+h_2+\cdots] \rangle = \left\langle p_\lambda, \prod_{k\geq 1}\prod_{d\mid k}\exp\left(d m_d \cdot \frac{p_k}{k}\right)\right\rangle$$
$$= \left\langle p_\lambda, \prod_{k\geq 1}\exp\left(\sum_{d\mid k} d m_d \frac{p_k}{k}\right)\right\rangle$$
$$= \left\langle p_\lambda, \prod_{k\geq 1}\sum_{r\geq 0}\frac{\left(\sum_{d\mid k} d m_d\right)^r p_k^r}{r!\,k^r}\right\rangle$$
$$= z_\lambda \cdot \prod_{k\geq 1}\frac{\left(\sum_{d\mid k} d m_d\right)^{\alpha_k}}{\alpha_k!\cdot k^{\alpha_k}} \qquad \text{where } \alpha_k = \#\, k \text{ in } \lambda$$
$$= \cancel{z_\lambda}\cdot\frac{\prod_{k\geq 1}\left(\sum_{d\mid k} d m_d\right)^{\alpha_k}}{\cancel{z_\lambda}}$$
$$= \prod_{k\geq 1}\left(\sum_{d\mid k} d m_d\right)^{\alpha_k}$$
$$= \prod_{i=1}^{\ell(\lambda)}\left(\sum_{d\mid \lambda_i} d m_d\right)$$

$$\langle \phi_n(p_\lambda), p_\mu \rangle = z_\mu \cdot \frac{p_\lambda[\Xi_\mu]}{z_\mu}$$
$$= p_\lambda[\Xi_\mu]$$
$$= \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[\Xi_\mu]$$
$$= \prod_{i=1}^{\ell(\lambda)}\left(\sum_{d\mid\lambda_i} d m_d\right)$$
$$= \langle p_\lambda, p_\mu[1+h_1+h_2+\cdots] \rangle.$$

$\square$

Lemma: (Character polynomials)

For every $f \in \Lambda$ with $\deg(f) \le n$, there exists a polynomial $g(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ s.t. ← character polynomial

$$f[\Xi_\mu] = g(m_1, m_2, \dots, m_n) \qquad \forall \mu = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \in \mathrm{Par}.$$

Proof: Let $f \in \Lambda$ with $\deg(f) \le n$.

Define $g(x_1, \dots, x_n) := f\Big|_{p_k \mapsto \sum_{d|k} d x_d}$

$$\therefore \text{ If } f = \sum_\lambda c_\lambda p_\lambda, \text{ then } f[\Xi_\mu] = \sum_\lambda c_\lambda p_\lambda[\Xi_\lambda] = \sum_\lambda c_\lambda \prod_{i=1}^{\ell(\lambda)} \underbrace{\sum_{d|\lambda_i} d m_d}_{p_{\lambda_i} \mapsto \sum_{d|\lambda_i} d m_d} = g(m_1, m_2, \dots, m_n). \qquad \square$$

$\therefore\ p_\lambda \mapsto \prod_{i=1}^{\ell(\lambda)} \sum_{d|\lambda_i} d m_d$

$f\Big|_{p_k \mapsto \sum_{d|k} d m_k}$

Given a character polynomial $g \in \mathbb{Q}[x_1, \dots, x_n]$, we can recover its corresponding character symmetric function $f \in \Lambda$ by:

$$f = g\left(p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \dots, \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) p_d\right)$$

Möbius formula: $\mu(r) =$ sum of the primitive r^{th} root of unity

$$= \begin{cases} (-1)^{\#\text{prime factors}} & \text{if } r \text{ is square free} \\ 0 & \text{otherwise} \end{cases}$$

e.g. $\mu(1) = 1$
$\mu(2) = -1$
$\mu(3) = -1$

Lemma: Let $f, g \in \Lambda$ s.t. $\deg(f), \deg(g) \le d$. Then $f = g$ iff $f[\Xi_\mu] = g[\Xi_\mu]$ $\forall |\mu| \le d$. (Property 1)

Proof: If $p(x)$ is a polynomial in one variable with $\deg(p(x)) \le d$. Then $p(x) = 0$ iff $p(0) = p(1) = \cdots = p(d) = 0$.

Multivariable version: (Prove by induction on # variables and degree)

$$p(x_1, \dots, x_d) = 0 \text{ iff } p(m_1, m_2, \dots, m_d) = 0 \quad \forall m_1 + 2m_2 + \cdots + d m_d \le d \quad (m_i \ge 0 \ \forall 1 \le i \le d).$$

(If we view x_i as x^i, then $p(x_1, \dots, x_d) = p(x, x^2, \dots, x^d)$ which reduces to the single variable case)

Let $h = f - g$ and $q(x_1, \dots, x_d)$ be the corresponding character polynomial.
i.e. $q(m_1, \dots, m_d) = h[\Xi_\mu]$, where $\mu = (1^{m_1} 2^{m_2} \cdots d^{m_d})$

$$\therefore f[\Xi_\mu] = g[\Xi_\mu] \ \forall |\mu| \le d \iff h[\Xi_\mu] = 0 \ \forall |\mu| \le d \iff q(m_1, \dots, m_d) = 0 \ \forall m_1 + 2m_2 + \cdots + d m_d \le d \iff q \equiv 0.$$

i.e. $f[\Xi_\mu] = g[\Xi_\mu]$ $\forall |\mu| \le d \iff q \equiv 0 \iff h \equiv 0 \iff f - g = 0 \iff f = g.$

Lemma: Let $f, g \in \Lambda$ s.t. $\deg(f), \deg(g) \leq d$. Then $f = g$ iff $f[\Xi_\mu] = g[\Xi_\mu]$ $\forall |\mu| > d$. (Property 2)

Proof: Note that $f[\Xi_{(c,\mu)}] = f[\Xi_\mu]$ if $c > d$ (b/c $p_k[\Xi_\gamma] = \sum_{r|k} r m_r$ where $m_r = \#r$ in γ.

$\because$ $k \leq d$ $\forall k$ appearing in p_k of f as $\deg(f) \leq d$

$\therefore$ $c \nmid k$ if $c > d$ ($\because c > d \Rightarrow c > k$)

$\therefore$ $p_k[\Xi_{(c,\gamma)}] = p_k[\Xi_\gamma]$ $\forall c > d$)

If $f[\Xi_\mu] = g[\Xi_\mu]$ $\forall |\mu| > d$, then pick $\mu = (d+1, \gamma)$ for some $\gamma \in \mathrm{Par}$ s.t. $|\gamma| \leq d$, we get

$$f[\Xi_{(d+1,\gamma)}] = g[\Xi_{(d+1,\gamma)}]$$

$$\therefore \quad f[\Xi_\gamma] = g[\Xi_\gamma]$$

$\because$ $|\gamma| \leq d$

$$\therefore \quad f = g \quad \text{(by property 1)}$$

(of cource if $f = g$ then $f[\Xi_\mu] = g[\Xi_\mu]$:)). $\square$

Let $b_\mu(x_1, \ldots, x_n) = 1^{a_1}(x_1)_{a_1}\, 2^{a_2}(x_2)_{a_2} \cdots n^{a_n}(x_n)_{a_n}$ where $\mu = 1^{a_1} 2^{a_2} \cdots n^{a_n}$ and $(x)_k = x(x-1)\cdots(x-k+1)$

e.g. $b_{32211}(x_1, x_2, x_3, x_4) = 1^2 (x_1)_2\, 2^2 (x_2)_2\, 3^1 (x_3)_1 \cdot 4^0 (x_4)_0$

$$= 12\, x_1(x_1-1) \cdot x_2(x_2-1) \cdot x_3$$

$$= 12 x_1 x_2 x_3 (x_1-1)(x_2-1)$$

$\therefore$ $b_\mu(a_1, \ldots, a_n) = 1^{a_1} \cdot a_1! \cdot 2^{a_2} \cdot a_2! \cdots n^{a_n} \cdot a_n! = z_\mu$.

Then $b_\mu(m_1, m_2, \ldots, m_n) = 1^{a_1} \cdot (m_1)_{a_1}\, 2^{a_2}(m_2)_{a_2} \cdots n^{a_n}(m_n)_{a_n}$

If $m_i < a_i$ for some i, then $(m_i)_{a_i} = 0$ and hence $b_\mu(m_1, \ldots, m_n) = 0$

If $m_i \geq a_i$ for all i, then $(m_i)_{a_i} = \frac{(m_i)_{m_i}}{(m_i - a_i)!} = \frac{m_i!}{(m_i - a_i)!}$, hence $b_\mu(m_1, \ldots, m_n) = \frac{1^{m_1} \cdot m_1! \cdot 2^{m_2} \cdot m_2! \cdots n^{m_n} \cdot m_n!}{1^{m_1 - a_1} (m_1 - a_1)! \cdot 2^{m_2 - a_2}(m_2 - a_2)! \cdots n^{m_n - a_n} (m_n - a_n)!}$

Define $\gamma = (1^{m_1}, \ldots, n^{m_n})$

$$= \frac{z_\gamma}{z_\tau} \quad \text{where } \tau = (1^{m_1 - a_1}, 2^{m_2 - a_2}, \ldots, n^{m_n - a_n})$$

(i.e. $p_\gamma = p_\mu p_\tau$)

i.e. $b_\mu(m_1, m_2, \ldots, m_n) = \begin{cases} \frac{z_\gamma}{z_\tau} & \text{if } m_i \geq a_i \ \forall i \in [n] \quad (\gamma = (1^{m_1} \cdots n^{m_n}),\ \tau = (1^{m_1 - a_1}, \ldots, n^{m_n - a_n})) \\ 0 & \text{otherwise} \end{cases}$

Define $\tilde{p}_\mu$ to be the character symmetric function whose character polynomial is b_μ.

i.e. $\tilde{p}_\mu = b_\mu\left(p_1, \frac{p_2-p_1}{2}, \frac{p_3-p_1}{3}, \dots, \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)p_d\right)$

Define $\tilde{h}_\mu = \sum_{\gamma\vdash|\mu|}\langle h_\mu, p_\gamma\rangle \frac{\tilde{p}_\gamma}{z_\gamma}$ and $\tilde{X}_\mu = \sum_{\gamma\vdash|\mu|}\langle s_\mu, p_\gamma\rangle\cdot\frac{\tilde{p}_\gamma}{z_\gamma}$.

Proposition: $\phi_n(\tilde{p}_\mu) = p_\mu h_{n-|\mu|}$ (for n: big)

$\phi_n(\tilde{h}_\mu) = h_\mu h_{n-|\mu|}$

$\phi_n(\tilde{X}_\mu) = s_\mu h_{n-|\mu|}$

Proof: $\phi_n(\tilde{p}_\mu) = \sum_{\gamma\vdash n}\tilde{p}_\mu[\Xi_\gamma]\cdot\frac{p_\gamma}{z_\gamma} = \sum_{\gamma\vdash n} b_\mu(m_1(\gamma), m_2(\gamma), \dots, m_n(\gamma))\cdot\frac{p_\gamma}{z_\gamma}$ where $m_i(\gamma) = \#i$ in γ

$$= \sum_{\gamma\vdash n}\frac{\cancel{z_\gamma}}{z_{\tau(\gamma)}}\cdot\frac{p_\gamma}{\cancel{z_\gamma}}$$ where $\tau(\gamma) = \left(1^{m_1(\gamma)-m_1(\mu)}, 2^{m_2(\gamma)-m_2(\mu)}, \dots, n^{m_n(\gamma)-m_n(\mu)}\right)$

$$= \sum_{\gamma\vdash n}\frac{p_\mu p_{\tau(\gamma)}}{z_{\tau(\gamma)}}$$

$$= \left(\sum_{\gamma\vdash n}\frac{p_{\tau(\gamma)}}{z_{\tau(\gamma)}}\right)p_\mu$$

$$= h_{|\tau(\gamma)|}p_\mu$$

$$= h_{|\gamma|-|\mu|}p_\mu$$

$$= h_{n-|\mu|}p_\mu.$$

$$\phi_n(\tilde{h}_\mu) = \sum_{\gamma\vdash|\mu|}\langle h_\mu, p_\gamma\rangle\cdot\frac{\phi_n(\tilde{p}_\gamma)}{z_\gamma}$$

$$= \sum_{\gamma\vdash|\mu|}\langle h_\mu, p_\gamma\rangle\cdot\frac{h_{n-|\gamma|}p_\gamma}{z_\gamma}$$

$$= \left(\sum_{\gamma\vdash|\mu|}\langle h_\mu, p_\gamma\rangle\cdot\frac{p_\gamma}{z_\gamma}\right)\cdot h_{n-|\mu|}$$ (b/c $|\gamma| = |\mu|$)

$$= h_\mu h_{n-|\mu|}$$

$$\phi_n(\tilde{X}_\mu) = \sum_{\gamma\vdash|\mu|}\langle s_\mu, p_\gamma\rangle\cdot\frac{\phi_n(\tilde{p}_\gamma)}{z_\gamma}$$

$$= \sum_{\gamma\vdash|\mu|}\langle s_\mu, p_\gamma\rangle\frac{h_{n-|\gamma|}p_\gamma}{z_\gamma}$$

$$= \left(\sum_{\gamma\vdash|\mu|}\langle s_\mu, p_\gamma\rangle\cdot\frac{p_\gamma}{z_\gamma}\right)\cdot h_{n-|\mu|}$$

$$= s_\mu h_{n-|\mu|}. \quad \square$$

Proposition: $\tilde{X}_\lambda = \sum_{\mu/\bar{\lambda}:\ \text{horizontal strips}}\tilde{s}_\mu$, $\quad\tilde{s}_\lambda = \sum_{\lambda/\mu:\ \text{vertical strips}}(-1)^{|\lambda|-|\mu|}\tilde{X}_\mu$. ← See Remark 17 in the paper (and I.G.Mac: P.124 (5))

($\bar{\lambda} = (\lambda_2, \lambda_3, \dots)$)

Proof: $\phi_n(\tilde{X}_\lambda) = s_\lambda h_{n-|\lambda|}$ for n big

$$= \sum_{\substack{\gamma/\lambda \\ \text{horizontal strips},\\ |\gamma|=n}} s_\gamma = \sum_{\mu/(\lambda_2,\dots,\lambda_{\ell(\lambda)})\text{ is a horizontal strip}} s_{n-|\mu|,\mu} = \sum_{\mu/\bar{\lambda}\text{ is a hori. strip}}\phi_n(\tilde{s}_\mu)$$ for any big n.

since n is big γ_1 is very "long"

we can consider $\mu = (\gamma_2, \gamma_3, \dots)$

$\therefore \gamma_1 = n - |\mu|$

$$\therefore \tilde{X}_\lambda = \sum_{\mu/\bar{\lambda}:\ \text{hori. strip}}\tilde{s}_\mu.$$

Proposition: $\tilde{h}_\mu = \sum_{|\lambda| \le |\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}\, \tilde{s}_\lambda$

Proof: $\varphi_n(\tilde{h}_\mu) = h_\mu h_{n-|\mu|}$

$$\sum_{|\lambda|\le|\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}\, \varphi_n(\tilde{s}_\lambda) = \sum_{|\lambda|\le|\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}\, s_{n-|\lambda|,\lambda} = h_{(n-|\mu|,\mu)} = h_{n-|\mu|} h_\mu = \varphi_n(\tilde{h}_\mu) \quad \forall \text{ big } n.$$

$$\therefore\ \tilde{h}_\mu = \sum_{|\lambda|\le|\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}\, \tilde{s}_\lambda.$$

$\square$

Theorem: $h_\lambda = \sum_{\substack{\pi:\ \text{multiset} \\ \text{partitions} \\ \text{with content } \lambda}} \tilde{h}_{m(\pi)}$, $m(\pi)$ = partition representing the multiplicities of the parts of π.

$$\tilde{h}_\mu = \sum_{T:\ \text{Tab.}} \tilde{s}_{w(T)}$$

multiset tab.

weight of Tab

e.g. $\lambda = (2,2)$ $\quad \therefore\ \pi$ has "1, 1, 2, 2".

π	$m(\pi)$
1|1|2|2	(2,2)
11|2|2	(2,1)
1|1|22	(2,1)
1|2|12	(1,1,1)
1|122	(1,1)
2|112	(1,1)
11|22	(1,1)
12|12	(2)
1122	(1)

$$\therefore\ h_{22} = \tilde{h}_{22} + 2\tilde{h}_{21} + \tilde{h}_{111} + 3\tilde{h}_{11} + \tilde{h}_2 + \tilde{h}_1$$