Symmetric Group Characters as Symmetric Functions

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ABSTRACT: Introduce non-homogeneous bases of Symmetric Functions s.t.

1. They evaluate to characters of the symmetric group.
2. Their structure coefficients correspond to the Kronecker product under regular product of polynomials.

We will also give applications to finding special cases of these products.

Assumption: Familiarity with rep. Theory of S_n.

Thanks: Anna and Jim for organizing and inviting us.

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Lect 1: Symmetric Polynomials as characters of GL_n

Goal for today: Motivation and Preliminaries.

1. Classical Schur-Weyl duality
2. Kronecker + Reduced Kronecker
3. Partition Algebra.

GL_n — $n \times n$ matrices

$GL(V)$ — V: dim n

Polynomial Reps of GL_n

$$\rho : GL_n \to GL_N$$

homomorphism

or W - vector space w/ an action of GL_n

Examples: 1) $\rho : GL_2 \to GL_3$

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$

$W = Sym^2(\mathbb{C}^2)$

2) Defining rep: $\varphi : GL_n \to GL_n$

$\varphi(A) = A$

$W = \mathbb{C}^n$.

A polynom. rep ρ is homog. of degree m if all entries in $\rho(A)$ are hom. of degree m.

Prop: $\rho : GL_n \to GL_N$ hom. poly of deegree m. then there exist a multiset

$$\mathcal{M}_\rho = \{ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha_1 + \cdots + \alpha = m \}$$

Containing exactly N monomials, s.t. if $A \in GL_n$ w/ eigenvalues $\theta_1, \ldots, \theta_n$, then $\rho(A)$ has eigenvalues θ^α, for all $x^\alpha \in \mathcal{M}_\rho$.

Examples: 1) $Sym^2(\mathbb{C}^2)$ $\quad \mathcal{M}_\rho = \{ x_1^2, x_1x_2, x_2^2 \}$

2) $V = \mathbb{C}^n$ $\quad \mathcal{M}_\rho = \{ x_1, \ldots, x_n \}$

Characters of GL_n:

Def: $\rho : GL_n \to GL_N$, $\quad \chi^\rho : GL_n \to \mathbb{C}$.

$$\chi^\rho(A) = \text{trace}(\rho(A))$$
$$= \text{sum of the diagonal entries in } \rho(A)$$
$$= \text{Sum of the eigenvalues of } \rho(A).$$

Examples: 1) $\chi^{\rho}(A) = \theta_1^2 + \theta_1\theta_2 + \theta_2^2$

2) $\chi^{\varphi}(A) = \theta_1 + \theta_2 + \cdots + \theta_n$

In general,

$$(*) \qquad \chi^{\rho} = \sum_{\chi^{\alpha} \in \mathcal{M}_{\rho}} \chi^{\alpha} \qquad \text{homogeneous polynomial.}$$

Note: To find the character value of $A \in Gl_n$ you evalute the polynomial $(*)$ at the eigenvalues of A.

Thm: 1) Every poly. rep of Gl_n is the direct sum of irreducibles.

2) Two poly. reps of Gl_n are isom. iff their characters are the same.

3) The irred. poly. rep. of Gl_n are indexed by partitions of length $\leq n$

(*) 4) The character of an irred. polynomial rep of $\underline{\underline{Gl_n}}$ is $\chi^{\lambda} = S_{\lambda}(x_1, \dots, x_n)$

How is this Theorem proved? SW-duality.

Idea:

- Let $V = \mathbb{C}^n$, $V^{\otimes k}$ k-fold tensor product.

(1) Gl_n acts diagonally on $V^{\otimes k}$

$$A \cdot (v_1 \otimes \cdots \otimes v_k) = Av_1 \otimes \cdots \otimes Av_k, \quad A \in Gl_n$$

(2) S_k acts on $V^{\otimes k}$ by permuting factors

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)} \qquad \sigma \in S_k$$

Key: These two actions commute.

$\Rightarrow$ $S_k \times GL_n$ acts on $V^{\otimes k}$

Critical idea: The actions "centralize" each other.

$n \geq 2k$

$End_{GL_n}(V^{\otimes k})$ = set of lin. transf $f: V^{\otimes k} \to V^{\otimes k}$ that commute with the action (diag) of GL_n on $V^{\otimes k}$

$\cong \mathbb{C}S_k$ (where lin. transf. arising from perm. action)

$End_{S_k}(V^{\otimes k})$ = generated by diagonal action of GL_n on $V^{\otimes k}$.

We have $V^{\otimes k}$ is an $S_k \times GL_n$ rep.

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \mathbb{S}^\lambda \otimes \mathbb{V}^\lambda$$

irred. rep of S_k

irred. poly reps deg k of GL_n

Compute the character of $V^{\otimes k}$:

$(\sigma, A) \in S_k \times GL_n$

$tr(\sigma, A)$ = trace of lin. transf. resulting from the action of $\sigma \times A$ on $V^{\otimes k}$.

$$= \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \chi^\lambda_{S_k}(\sigma)\, \mathrm{trace}(\rho^\lambda(A))$$

It can be shown: σ has cycle type μ

$$\mathrm{trace}(\sigma, A) = P_\mu(\underbrace{\theta_1, \ldots, \theta_n}_{\text{eigenvalues of } A})$$

<u>Example</u>: $n = k = 2$

$$\left((12), \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right) \in S_2 \times Gl_2$$

$V = \mathbb{C}^2$ and $V^{\otimes 2} = \mathrm{Span}\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$

$$\begin{bmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & 0 & x_1 x_2 & 0 \\ 0 & x_1 x_2 & 0 & 0 \\ 0 & 0 & 0 & x_2^2 \end{bmatrix}$$

$$\mathrm{trace} = x_1^2 + x_2^2 = P_2(x_1, x_2)$$

$$P_\mu(\theta_1, \ldots, \theta_n) = \sum_\lambda \chi^\lambda(\sigma)\, \underline{\mathrm{trace}(\rho^\lambda(A))}$$

MN-Rule

$$P_\mu(\theta_1, \ldots, \theta_n) = \sum_\lambda \chi^\lambda(\sigma)\, S_\lambda(\theta_1, \ldots, \theta_n)$$

$\{\chi^\lambda\}$: lin. independent from Rep. Th. of Symmetric group.

<u>MAIN FACT 1</u>: Irred. characters of Gl_n are Schur polynom. evaluated at eigenvalues of $A \in Gl_n$.

<u>Tensor Products</u>: $\mathbb{V}^\lambda, \mathbb{V}^\mu$ are irred. poly. reps of Gl_n

$\mathbb{V}^\lambda \otimes \mathbb{V}^\mu$ w/ diagonal action.

Character of this representation

$$S_\lambda(\theta_1, \ldots, \theta_n)\, S_\mu(\theta_1, \ldots, \theta_n)$$

$$\mathbb{V}^\lambda \otimes \mathbb{V}^\mu = \bigoplus_\nu (\mathbb{V}^\nu)^{\oplus c^\nu_{\lambda\mu}}$$

Littlewood-Richardson coefficients.

Main Fact 2: Decomposing tensors of $\overset{\text{poly.}}{}$irred reps of GL_n

$\Updownarrow$ corresp.

Products of Schur functions.

Kronecker Coefficients for S_n $g^{\nu}_{\lambda\mu}$ $''$

$$S^{\lambda} \otimes S^{\mu} = \bigoplus_{\nu} \left(S^{\nu} \right)^{\oplus \underbrace{g(\lambda,\mu,\nu)}_{\text{Kronecker coefficients}}}$$

$\lambda, \mu, \nu \vdash n$

Example: $S^{(2,2)} \otimes S^{(2,2)} = S^{(4)} \oplus S^{(1,1,1,1)} \oplus S^{(2,2)}$

	(1)(2)(3)(4)	(12)(3)(4)	(123)(4)	(12)(34)	(1234)
$\chi^{(1,1,1,1)}$	1	−1	1	1	−1
$\chi^{(2,1,1)}$	3	−1	0	−1	1
$\chi^{(2,2)}$	2	0	−1	2	0
$\chi^{(3,1)}$	3	1	0	−1	−1
$\chi^{(4)}$	1	1	1	1	1

$$\chi^{(2,2)} = \langle 2, 0, -1, 2, 0 \rangle$$

$$\chi^{(2,2)} \cdot \chi^{(2,2)} = \langle 4, 0, 1, 4, 0 \rangle$$

$$= \langle 1, 1, 1, 1, 1 \rangle + \langle 1, -1, 1, 1, -1 \rangle + \langle 2, 0, -1, 2, 0 \rangle$$

Murnagham-Stability:

$$g((\lambda_1, \lambda_2, \ldots), (\mu_1, \mu_2, \ldots), (\nu_1, \nu_2, \ldots))$$

$$\leq g(\lambda_1 + 1, \lambda_2), (\mu_1 + 1, \mu_2, \ldots), (\nu_1 + 1, \nu_2, \ldots)$$

Eventually it becomes constant.

$\chi^{\lambda} \chi^{\mu}$

Example: [BOR] $|\bar\lambda| + |\bar\mu| + \lambda_2 + \mu_2$

$\chi^{2,2}\chi^{2,2} = \chi^{4} + \chi^{1,1,1,1} + \chi^{2,2}$

$\chi^{3,2}\chi^{3,2} = \chi^{5} + \chi^{2,1,1,1} + \chi^{3,2} + \chi^{4,1} + \chi^{3,1,1} + \chi^{2,2,1}$

$\chi^{4,2}\chi^{4,2} = \chi^{6} + \chi^{3,1,1,1} + 2\chi^{4,2} + \chi^{5,1} + \chi^{4,1,1} + 2\chi^{3,2,1} + \chi^{2,2,2}$

$\chi^{5,2}\chi^{5,2} = \chi^{7} + \chi^{4,1,1,1} + 2\chi^{5,2} + \chi^{6,1} + \chi^{5,1,1} + 2\chi^{4,2,1} + \chi^{3,2,2} + \chi^{4,3} + \chi^{3,3,1}$

$\chi^{6,2}\chi^{6,2} = \chi^{8} + \chi^{5,1,1,1} + 2\chi^{6,2} + \chi^{7,1} + \chi^{6,1,1} + 2\chi^{5,2,1} + \chi^{4,2,2} + \chi^{5,3} + \chi^{4,3,1} + \chi^{4,4}$

$\chi^{7,2}\chi^{7,2} = \chi^{9} + \chi^{6,1,1,1} + 2\chi^{7,2} + \chi^{8,1} + \chi^{7,1,1} + 2\chi^{6,2,1} + \chi^{5,2,2} + \chi^{6,3} + \chi^{5,3,1} + \chi^{5,4}$

$\chi^{\bullet,2}\chi^{\bullet,2} = \chi^{\bullet} + \chi^{\bullet,1,1,1} + 2\chi^{\bullet,2} + \chi^{\bullet,1} + \chi^{\bullet,1,1} + 2\chi^{\bullet,2,1} + \chi^{\bullet,2,2} + \chi^{\bullet,3} + \chi^{\bullet,3,1} + \chi^{\bullet,4}$

$\bar{g}_{\alpha,\beta}^{\gamma}$ stable limits of the sequences.
"reduced" (stable) Kronecker coefficients.

Thm: [BOR]

$$g_{\lambda\mu}^{\nu} = \sum_{i=1}^{\ell(\lambda)\ell(\mu)} (-1)^{i+1}\, \bar{g}_{\bar\lambda\,\bar\mu}^{\,\nu+i} \qquad \bar\lambda = (\lambda_2, \dots)$$

$u = (u_1, u_2, \dots)$

remove

$u^{+i} = (u_1+1, \dots, u_{i-1}+1, \widehat{u_i}, u_{i+1}, \dots)$

Main Problem: Find a combinatorial formulation for $g_{\lambda\mu}^{\nu}$ or $\bar{g}_{\alpha\beta}^{\gamma}$.

IDEA: Think $S_n \subseteq GL_n$

$n\times n$ permutation matrices.

"replicate" what worked for GL_n.

Restriction Approach:

1) Restrict SW Duality. $S_n \subseteq Gl_n$

- S_n acts diagonally on $V^{\otimes k}$

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \sigma.v_1 \otimes \cdots \otimes \sigma(v_k)$$

What commutes with this action? $(12)\, e_1 \otimes e_1 = e_2 \otimes e_2$

$$\mathrm{End}_{S_n}(V^{\otimes k}) \cong P_k(n)$$

$n \geq 2k$ Partition algebra

If a $n^k \times n^k$ matrix A commutes with the action of S_n

$$A = \left(A^{i_1, \ldots, i_k}_{j_1, \ldots, j_k} \right) \quad i \leftarrow \{1, 2, \ldots, k\}, \quad j \leftarrow \{\bar{1}, \bar{2}, \ldots, \bar{k}\}$$

<u>Lemma</u>: $A \in \mathrm{End}_{S_n}(V^{\otimes k}) \iff A^{\vec{i}}_{\vec{j}} = A^{\sigma(\vec{i})}_{\sigma(\vec{j})}$

$$\sigma(\vec{i}) = (\sigma(i_1), \sigma(i_2), \ldots \quad)$$

<u>Example</u>: $n = k = 2$

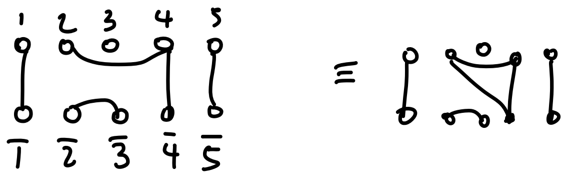
$$A = \begin{bmatrix} A^{11}_{11} & A^{12}_{11} & A^{21}_{11} & A^{22}_{11} \\ A^{11}_{12} & \ddots & & \\ \vdots & & & \end{bmatrix}$$

$A^{11}_{11} = A^{22}_{22}$, $A^{12}_{12} = A^{22}_{21}$, $A^{12}_{11} = A^{21}_{22}$, etc.

$\{1, 2, \bar{1}, \bar{2}\}$ $\quad \{\bar{1}, 1\}\{2, \bar{2}\}$ $\quad \{\bar{1}, 1, \bar{2}\}\{2\}$

The set partitions of $[k] \cup [\bar{k}]$ index the lin. ind. linear transf. that commute w/ the diagonal action of S_n

$P_K(n) = \mathbb{C}(n)\text{-Span}\{\text{set partitions of } [K] \cup [\bar{K}]\}$

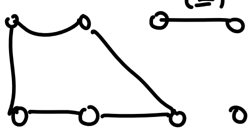


Connectivity.

$\{1,\bar{1}\}, \{2,4,\bar{4}\}, \{\bar{2},\bar{3}\}, \{5,\bar{5}\}\{3\}$

$\delta_{a=b=c}\,(v_a \otimes v_a) \otimes \sum_{i=1}^{n} v_i \otimes v_i$

$f \uparrow$



$v_a \otimes v_b \otimes v_c \otimes v_d$

$n=2$

$f(v_1 \otimes v_2 \otimes v_1 \otimes v_2) = 0$

$f(v_1 \otimes v_1 \otimes v_1 \otimes v_2)$

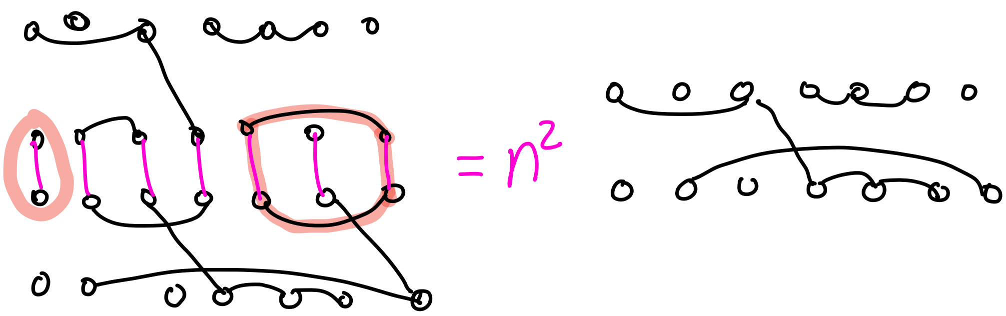
$= v_1 \otimes v_1 \otimes v_1 \otimes v_1$

$+ v_1 \otimes v_1 \otimes v_2 \otimes v_2$

<u>Product</u>:

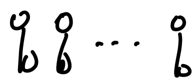
d_1

$d_1 d_2 =$



$= n^2$

Assoc., $\dim P_K(n) = B(2K)$, identity



<u>SW-duality for S_n - $P_K(n)$</u>

- Both act on $V^{\otimes K}$
- Actions Commute

$\Rightarrow$ $\underline{P_K(n)} \times S_K$ acts on $V^{\otimes K}$

$\boxed{n \geq 2K}$

$$\Rightarrow \quad V^{\otimes K} \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_2+\cdots+\lambda_\ell \leq K}} (\mathbb{L}^\lambda \otimes S^\lambda)$$

irred. rep of $P_K(n)$

Compute character. $(d, \sigma) \in P_k(n) \times S_n$

$$\mathrm{Char}(V^{\otimes k}) = \sum \chi^{\lambda}_{P_n(n)}(d)\, \chi^{\lambda}_{S_n}(\sigma)$$

$S_n \subseteq GL_n$

$$\mathrm{Char}(V^{\otimes k}) = p_\mu(\underbrace{x_1, \ldots, x_n}_{\substack{\text{evaluating at} \\ \text{eigvalutig of permutation} \\ \text{matrices}}})$$

<u>Conj:</u>

$$p_\mu(x_1, \ldots, x_n) = \sum_{\lambda} \chi_{P_k(n)}(d)\, \underbrace{\tilde{S}_{\bar{\lambda}}(x_1, \ldots, x_n)}$$

$x_1, \ldots, x_n$ are eigenvalues of $\sigma \in S_n$.

Evaluates to irreducibles of S_n

Contrast.

$$p_\mu = \sum \chi_{S_k}(\sigma)\, s_\lambda(x_1, \ldots, x_n)$$

Furthermore, $\underbrace{\tilde{S}_\lambda \tilde{S}_\mu}_{\text{product of poly.}} = \sum \bar{g}^{\nu}_{\lambda\mu} \tilde{S}_\gamma$