Symmetric group characters as symmetric functions

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1. Question of Rosa: is there a basis of the symmetric functions that has as structure coefficients the reduced Kronecker coefficients?

$$b_\lambda \cdot b_\mu = \sum_\gamma \bar{g}^{\gamma}_{\lambda\mu} b_\gamma$$

2. modules and characters of S_n
3. Frobenius character map
4. Restriction problem: decompose an irred Gl_n-module into irred S_n-modules
5. Frobenius map and Kronecker product
6. Answer: definition of the character basis

An S_n module is vector space with S_n action

Examples:

$V_1 = \text{span}\{v\}$

$\sigma \in S_n \quad \sigma \cdot v = v$

$V_2 = \text{span}\{w\}$

$\sigma \in S_n \quad \sigma \cdot w = \text{sgn}(\sigma) w$

$V_3 = \text{span}\{v_1, v_2, \ldots, v_n\}$

$\sigma \in S_n \quad \sigma \cdot v_i = v_{\sigma(i)}$

$V_4 = \text{span}\{v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} : a_1 + a_2 + \cdots + a_n = d\}$
$= \text{Sym}^d(\mathbb{C}^n)$

$\sigma \cdot v_1^{a_1} \cdots v_n^{a_n} = v_{\sigma(1)}^{a_1} \cdots v_{\sigma(n)}^{a_n}$

A character of an S_n-module V

basis $\mathcal{B}_V$

$\chi : S_n \to \mathbb{C} \qquad \sigma \in S_n$

$$\chi(\sigma) = \sum_{b \in \mathcal{B}_V} \text{coeff of } b \text{ in } \sigma \cdot b$$

Theorem: χ is a character for a S_n-module V

① if $\sigma, \tau \in S_n$ and σ and τ are in same conj class the $\chi(\sigma) = \chi(\tau)$

σ has cycle type μ $\quad cyc(\sigma) = \mu \quad \chi(\mu) := \chi(\sigma)$

② there are irreducible S_n modules S^λ indexed by partitions λ of n $(\lambda \vdash n)$. the characters of irred are denoted χ^λ

③ there are integers $m_\lambda(V) \in \mathbb{Z}_{\geq 0}$ for each $\lambda \vdash n$ s.t.

$$V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus m_\lambda(V)} \iff \chi = \sum_{\lambda \vdash n} m_\lambda(V) \chi^\lambda$$

④

$$s_\lambda = \sum_{\mu \vdash n} \chi^\lambda(\mu) \frac{p_\mu}{z_\mu} \qquad p_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda$$

$$z_\lambda := \prod_{i \geq 1} m_i(\lambda)!\, i^{m_i(\lambda)} \qquad m_i(\lambda) = \#\text{ of parts of size } i \text{ in } \lambda$$

5 modules are isomorphic iff their characters are equal

the characters "characterize" the module

	ϕ_1	ϕ_2	ϕ_3
$(1)(2)(3) \longrightarrow$	$\underset{3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$	$\underset{3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$	$\underset{3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$
$(1)(23) \longrightarrow$	$\underset{-1}{\begin{bmatrix} 1 & -6 & 6 \\ -1 & 2 & -3 \\ -1 & 3 & -4 \end{bmatrix}}$	$\underset{1}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 6 & -7 \\ -1 & 5 & -6 \end{bmatrix}}$	$\underset{1}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{bmatrix}}$
$(12)(3) \longrightarrow$	$\begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 7 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$(123) \longrightarrow$	$\underset{0}{\begin{bmatrix} 0 & 2 & -1 \\ -1 & 6 & 7 \\ -1 & 5 & -6 \end{bmatrix}}$	$\underset{0}{\begin{bmatrix} 0 & 2 & -1 \\ -1 & 6 & -7 \\ -1 & 5 & -6 \end{bmatrix}}$	$\underset{0}{\begin{bmatrix} 4 & 2 & 3 \\ -3 & -2 & -3 \\ -3 & -1 & -2 \end{bmatrix}}$
$(132) \longrightarrow$	$\begin{bmatrix} -1 & 7 & -8 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$	$\begin{bmatrix} -1 & 7 & -8 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 3 \\ -3 & -2 & -2 \end{bmatrix}$
$(13)(2) \longrightarrow$	$\begin{bmatrix} 0 & -4 & 5 \\ 1 & -5 & 5 \\ 1 & -4 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 4 & 2 & 3 \\ -3 & -1 & -3 \\ -3 & -2 & -2 \end{bmatrix}$

$$V_K = \operatorname{span}\{V_1, V_2, V_3\} \qquad K \in \{1,2,3\}$$

$$\sigma \cdot V_i = \sum_{j=1}^{3} \phi_K(\sigma)_{ij} V_j$$

$\phi_1 \ncong \phi_2$ or ϕ_3 || $V_1 \ncong V_2$ or V_3

$\phi_2 \cong \phi_3$ || $V_2 \cong V_3$

		$(1)(2)(3)$	$(13)(2)$, $(23)(1)$, $(12)(3)$	(132), (123)
irred char index	(3)	1	1	1
	(21)	2	0	-1
	(111)	1	-1	1
	$G\ \chi^{V_3}$	3	1	0

$$\chi^{V_3} = \chi^{(21)} + \chi^{(3)}$$

Define character map $\qquad \chi$ is an S_n-class function

$$\mathcal{F}_n(\chi) = \sum_{\mu \vdash n} \chi(\mu)\, p_\mu / z_\mu$$

$$\mathcal{F}_n(\chi^\lambda) = \sum_{\mu \vdash n} \chi^\lambda(\mu)\, p_\mu / z_\mu = s_\lambda$$

$$V \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus m_\lambda(V)} \iff \chi^V = \sum_{\lambda \vdash n} m_\lambda(V)\, \chi^\lambda$$

$$\iff \mathcal{F}_n(\chi^V) = \sum_{\lambda \vdash n} m_\lambda(V)\, s_\lambda$$

$\mathcal{F}_n$: class functions $\longrightarrow$ $\Lambda_n :=$ symmetric functions of degree n

Example: $V = \mathrm{Sym}^2(\mathbb{C}^2) = \mathrm{span}\{v_1^2, v_1v_2, v_2^2\}$

Gl_2 - character $= s_2(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2$

S_2 - character $\qquad S_2 \subseteq Gl_2$

$(1)(2)$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has e.v $1,1$ $\qquad \chi^V((1)(2)) = s_2(1,1) = 3$

(12) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has e.v. $1,-1$ $\qquad \chi^V((12)) = s_2(1,-1) = 1$

$$\mathcal{F}(\chi^V) = 3\,p_{11}/2 + 1\cdot p_2/2 = 2\cdot s_2 + s_{11}$$

$$V \cong S^{(2)} \oplus S^{(2)} \oplus S^{(11)}$$

$$d\begin{bmatrix} 0 & & & 0 & 1 \\ 1 & \ddots & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{bmatrix} \quad \text{has e.v.} \quad 1, \zeta_d, \zeta_d^2, \ldots, \zeta_d^{d-1} \quad \text{where } \zeta_d = e^{2\pi i/d}$$

$$\qquad d$$

$$p_k(1, \zeta_d, \zeta_d^2, \ldots, \zeta_d^{d-1}) = 1^k + \zeta_d^k + \zeta_d^{2k} + \cdots + \zeta_d^{k(d-1)} = \begin{cases} d & \text{if } d \mid k \\ 0 & \text{else} \end{cases}$$

$$\shortparallel$$

$$p_k[\Xi_d] \qquad\qquad \gamma = (1^{m_1} 2^{m_2} \ldots d^{m_d})$$

$$p_k[\Xi_\gamma] = p_k[\Xi_{\gamma_1}] + p_k[\Xi_{\gamma_2}] + \cdots + p_k[\Xi_{\gamma_{\ell(\gamma)}}]$$

$$= \sum_{r=1}^{\ell(\gamma)} p_k[\Xi_r] = \sum_{r=1}^{d} m_r p_k[\Xi_r]$$

$$= \sum_{r \mid k} r \cdot m_r$$

$$p_\mu[\Xi_\gamma] = p_{\mu_1}[\Xi_\gamma]\, p_{\mu_2}[\Xi_\gamma] \cdots p_{\mu_{\ell(\mu)}}[\Xi_\gamma]$$

$$\Phi_n : \Lambda \longrightarrow \Lambda_n \qquad \Lambda := \bigoplus_{n\geq 0} \Lambda_n$$

$$\Phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu]\, p_\mu / z_\mu$$

Example: $V = Sym^2(\mathbb{C}^3)$ is a Gl_3-module also S_3-module

Gl_3 character $s_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2$

(3)	1	1	1
(21)	2	0	-1
(111)	1	-1	1
χ^V	6	2	0

$1,1,1 \quad s_2[\Xi_{111}] = 6$

$1,-1,1 \quad s_2[\Xi_{21}] = 2$

$1,\zeta_3,\zeta_3^2 \quad s_2[\Xi_3] = 0$

$$\Phi_3(s_2) = 6 \cdot p_{111}/6 + 2 \cdot p_{21}/2$$
$$= 2s_3 + 2s_{21}$$

$$Sym^2(\mathbb{C}^3) \cong S^{(3)} \oplus S^{(3)} \oplus S^{(2,1)} \oplus S^{(2,1)}$$

$$S^{(2,2)} \otimes S^{(2,2)} \cong S^{(4)} \oplus S^{(2,2)} \oplus S^{(1111)}$$

more generally if char χ^V and χ^W are characters of V & W then character of $V \otimes W$ is $\chi^V \cdot \chi^W$

$$(\chi^V \chi^W)(\sigma) = \chi^V(\sigma)\chi^W(\sigma)$$

Kronecker product on symmetric functions

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda=\mu} \frac{p_\lambda}{z_\lambda} \qquad \mathcal{F}_n(\chi^V) * \mathcal{F}_n(\chi^W) = \mathcal{F}_n(\chi^V \cdot \chi^W)$$

$$\Phi_n(f) * \Phi_n(g) = \Phi_n(f \cdot g)$$

$$\left(\sum_{\mu \vdash n} f[\Xi_\mu] p_\mu / z_\mu\right) * \left(\sum_{\gamma \vdash n} g[\Xi_\gamma] \frac{p_\gamma}{z_\gamma}\right) = \sum_{\nu \vdash n} f[\Xi_\nu] g[\Xi_\nu] \frac{p_\nu}{z_\nu}$$

for n big

$$s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)} = \sum_{\gamma} \bar{g}^{\gamma}_{\lambda\mu}\, s_{(n-|\gamma|,\gamma)}$$

$$\tilde{s}_\lambda := \Phi_n^{-1}\left(s_{(n-|\lambda|,\lambda)}\right)$$

$$\tilde{s}_\lambda \cdot \tilde{s}_\mu = \sum_{\gamma} \bar{g}^{\gamma}_{\lambda\mu}\, \tilde{s}_\gamma$$

<u>iff</u>

$$\Phi_n(\tilde{s}_\lambda \cdot \tilde{s}_\mu) = \sum_{\gamma} \bar{g}^{\gamma}_{\lambda\mu}\, \Phi_n(\tilde{s}_\gamma)$$

$$\Phi_n(\tilde{s}_\lambda \cdot \tilde{s}_\mu) = \Phi_n(\tilde{s}_\lambda) * \Phi_n(\tilde{s}_\mu) = s_{(n-|\lambda|,\lambda)} * s_{(n-|\mu|,\mu)}$$

$$\sum_{\gamma} \bar{g}^{\gamma}_{\lambda\mu}\, \Phi_n(\tilde{s}_\gamma) = \sum_{\gamma} \bar{g}^{\gamma}_{\lambda\mu}\, s_{(n-|\gamma|,\gamma)}$$

S_2		
(2)	1	1
(11)	1	-1

$$\Phi_2(1) = p_{11}/2 + p_2/2 = s_{(2)}$$

$$\tilde{s}_{()} = 1$$

$$\Phi_2(s_1) = 2\,p_{11}/2 + 0\,p_2/2 = s_{(2)} + s_{(11)}$$

$$\Phi_2(s_1 - 1) = p_{11}/2 - p_2/2$$

$$\tilde{s}_{(1)} = s_1 - 1$$

	(1)(2)(3)	(23)(1) (13)(2) (12)(3)	(132) (123)
(3)	1	1	1
(2,1)	2	0	−1
(1,1,1)	1	−1	1

	(1111)	(211)	(22)	(31)	(4)
(4)	1	1	1	1	1
(3,1)	3	1	0	0	−1
(2,2)	2	0	2	−1	0
(2,1,1)	3	−1	0	0	1
(1,1,1,1)	1	−1	1	1	−1
$\chi^{(22)} \otimes \chi^{(22)}$	4	0	4	1	0