Symmetric Group Characters as Symmetric Functions

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ABSTRACT: Introduce non-homogeneous bases of Symmetric Functions s.t.

① They evaluate to characters of the symmetric group.

② Their structure coefficients correspond to the Kronecker product under regular product of polynomials.

We will also give applications to finding special cases of these products.

Assumption: Familiarity with rep. Theory of S_n.

Thanks: Anna and Jim for organizing and inviting us.

Lect 4: Change of basis coefficients. - Combinatorics

Recap: If $\sigma \in S_n$ has cycle type μ, its permutation matrix has eigenvalues

$$\Xi_\mu = \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_\ell}$$

where $\Xi_r = 1, \rho, \rho^2, \ldots, \rho^{r-1}$ and $\rho = e^{\frac{2\pi i}{r}}$

Mike: 1) If $f(x_1, \ldots, x_n) \in \Lambda_n$ (a GL_n-character)

$$\phi_n(f) = \sum_{\mu \vdash n} f(\Xi_\mu) \frac{p_\mu}{z_\mu}$$

$$\Rightarrow f(\Xi_\mu) = \langle \phi_n(f), p_\mu \rangle \quad (\text{a class fnc of } S_n)$$

example: $h_{1,1}(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

$$h_{1,1}(\Xi_{(1,1,1)}) = 9, \quad h_{1,1}(\Xi_{2,1}) = 1, \quad h_{1,1}(\Xi_3) = 0$$

2) $\phi_n(\tilde{p}_\mu) = p_\mu h_{n-|\mu|}$, $\phi_n(\tilde{h}_\mu) = h_{(n-|\mu|,\mu)}$, $\phi_n(\tilde{s}_\mu) = s_{(n-|\mu|,\mu)}$

3) Thms: $f, g \in \Lambda$ of degree $\leq d \in \mathbb{Z}_{>0}$

$$f = g \Leftrightarrow f(\Xi_\mu) = g(\Xi_\mu) \quad \forall |\mu| \leq d \quad \left(\begin{array}{c}\text{Also true if} \\ \forall |\mu| > d\end{array}\right).$$

Today's Goal: $h \to \tilde{h}$, $h \to \tilde{s}$, $e \to \tilde{s}$ n big

$$\phi_n(\tilde{h}_\mu) = h_{(n-|\mu|,\mu)}$$

$$\Rightarrow h_{(n-|\mu|,\mu)} = \sum_{\mu \vdash n} \tilde{h}_\gamma(\Xi_\mu) \frac{p_\mu}{z_\mu}$$

Rep. Th. S_n or Symm. fnc [SAGAN].

$\langle h_{(n-|\gamma|,\gamma)}, p_\mu \rangle$ = Character of the permutation module $M^{(n-|\gamma|,\gamma)}$ evaluated at σ w/ cycle type μ

- for any $\lambda \vdash n$, $M^\lambda \cong \mathbb{1}\big\uparrow_{S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}}^{S_n}$
- M^λ has a basis of tabloids.

Example: $n=3$

$M^{(3)} = \mathbb{C}\text{span}\{\overline{\underline{1\,2\,3}}\}$

$M^{(2,1)} = \mathbb{C}\text{-span}\left\{ \begin{smallmatrix} \overline{1\,2} \\ \overline{3} \end{smallmatrix}, \begin{smallmatrix} \overline{1\,3} \\ \overline{2} \end{smallmatrix}, \begin{smallmatrix} \overline{2\,3} \\ \overline{1} \end{smallmatrix} \right\}$

$M^{(1,1,1)} = \mathbb{C}\text{-Span}\left\{ \begin{smallmatrix} \overline{1} \\ \overline{2} \\ \overline{3} \end{smallmatrix}, \ldots \right\}$ 6 of them

Character: trace (σ) = # fixed tabloids by σ.

	$(1)(2)(3)$	(12)	(123)	
(3)	1	1	1	$\tilde{h}_\phi$
$(2,1)$	3	1	0	$\tilde{h}_1$
$(1,1,1)$	6	0	0	$\tilde{h}_{1,1}$

Definition: $\tilde{h}_\lambda[\Xi_\mu] := \langle h_{(n-|\lambda|,\lambda)}, p_\mu \rangle$

$$\boxed{\textbf{Thm: } n \text{ big}, \lambda \text{ any partition} \quad h_\lambda[\Xi_\mu] = \sum_{\pi \Vdash \{\{1^{\lambda_1}, \ldots, \ell^{\lambda_\ell}\}\}} \langle h_{(n-\ell(\pi), m(\pi))}, p_\mu \rangle}$$

Example: (Notation)

$\lambda = (2,1)$ $\quad \{\{1,1,2\}\} = M$

Multiset partitions of M:

$1/1/2$,	$1,1/2$,	$1,2/1$,	112
$m=(2,1)$	$m=(1,1)$	$m=(1,1)$	$m=(1)$
$\ell=3$	$\ell=2$	$\ell=2$	$\ell=1$

Cor: $$h_\lambda = \sum_{\pi \Vdash \{\{1^{\lambda_1},\dots\}\}} \tilde{h}_{m(\pi)}$$

ex: $$h_{2,1} = \tilde{h}_{(2,1)} + \tilde{h}_{(1,1)} + \tilde{h}_{(1,1)} + \tilde{h}_{(1)}$$
$$= \tilde{h}_{(2,1)} + 2\tilde{h}_{(1,1)} + \tilde{h}_{(1)}$$

Proof Thm: Note: RHS we sum only non-neg. integers

Show: Both sides count sets of the same size.

Lemma: (Lascoux) $r \geq 1$, $h_0[\Xi_r] = 1$
$$h_n[\Xi_r] = \delta_{r|n}$$

Prop: $h_n[\Xi_\mu]$ = # weak compositions α of n w/ $\ell(\mu)=\ell(\alpha)$ and s.t. $\mu_i \mid \alpha_i \ \forall i$

Example: $h_2[\Xi_{2,1}] = 2$ $\quad \mu = (2,1)$

$(2,0)$	$(0,2)$	$(1,1)$
$2\mid 2 \quad 1\mid 0$	$2\mid 0 \quad 1\mid 2$	$2\nmid 1$
✓	✓	✗

Pf/ $$h_n[X_1, X_2, \dots, X_r] = \sum_{\substack{\alpha \vDash_w n \\ \ell(\alpha)=r}} h_{\alpha_1}[X_1]\cdots h_{\alpha_r}[X_r]$$

Replace: $X_i = \Xi_{\mu_i}$
$$h_n[\Xi_\mu] = \sum_{\substack{\alpha \vDash_w n \\ \ell(\alpha)=\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \underbrace{h_{\alpha_i}[\Xi_{\mu_i}]}_{=\begin{cases}1 & \text{if } \mu_i \mid \alpha_i \\ 0 & \text{o.w.}\end{cases}}$$ ∎

$$C_{\lambda\mu} = \left\{ \left(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell(\lambda))}\right) : \begin{array}{l} \alpha^{(i)} \vDash_w \lambda_i \\ \ell(\alpha^{(i)}) = \ell(\mu) \\ \mu_j \mid \alpha_j^{(i)} \quad \forall j \end{array} \right\}$$

<u>Prop</u>: $h_\lambda[\Xi_\mu] = |C_{\lambda\mu}|$

Pf/ $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$, by prod. principle. ■

<u>In RHS of Thm</u>: We sum things that look like:

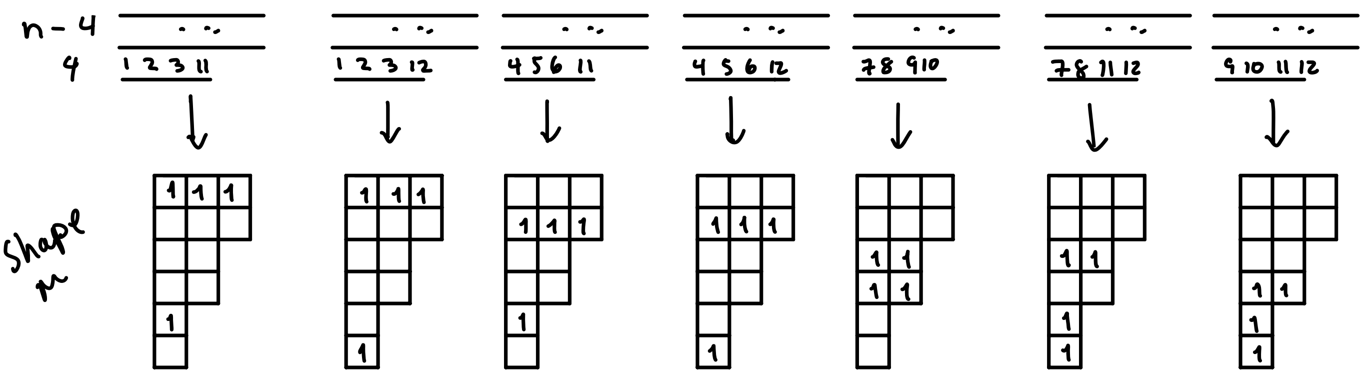
$\langle h_{(n-|\lambda|,\lambda)}, P_\mu \rangle$ = Character of $M^{(n-|\lambda|,\lambda)}$ at σ of cycle type μ

= # tabloids of shape $(n-|\lambda|,\lambda)$ that are fixed by this σ of type μ.

<u>Note</u>: If σ is in cycle notation, then σ fixes a tabloid iff each cycle permutes only elements in the same row.

<u>Example</u>: $\lambda = (4)$, $n = 12$ $\quad \mu = (3,3,2,2,1,1)$

$\sigma = (1,2,3)(4,5,6)(7,8)(9,10)(11)(12)$ $\quad (n-4, 4)$



<u>Prop</u>: $\langle h_{(n-|\lambda|,\lambda)}, P_\mu \rangle$ = # tableaux of shape μ where rows are filled w/ same # using $0, 1, 2, \dots, \ell(\lambda)$ and T has content $\{\{0^{n-|\lambda|}, 1^{\lambda_1}, \dots, \ell^{\lambda_\ell}\}\}$

<u>note</u>: we do not write the 0's.

RHS: $\sum_\pi \langle h_{(n-\ell(\pi), m(\pi))}, P_\mu \rangle$

Example: $\lambda = (12, 7, 2)$

$\pi = 1,2 \mid 1,2 \mid 1,2 \mid 1,1,1 \mid 1,1,1 \mid 1,2,2,3 \mid 1,2,2,3 \mid 1$

$m(\pi) = (3, 2, 2, 1)$

$\quad 1,2 \qquad 1,1,1 \qquad 1,2,2,3 \qquad 1$

$\langle h_{(n-8,\, m(\pi))}, P_\mu \rangle$ = # tableaux T of shape μ where $n-8$ boxes are filled $\emptyset$

3 are filled $\{\{1,2\}\}$

2 are filled $\{\{1,1,1\}\}$

2 are " $\{\{1,2,2,3\}\}$

1 " $\{\{1\}\}$

all rows have same multiset.

$\mathcal{T}_{\lambda,\mu} = \{$ tableau T of shape μ of this kind for $\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}\}$

$$\sum_{\pi} \langle h_{(n-\ell(\pi),\, m(\pi))}, P_\mu \rangle = |\mathcal{T}_{\lambda,\mu}|$$

Claim: There is a bijection $\mathcal{T}_{\lambda,\mu} \longrightarrow C_{\lambda,\mu}$

Pf/ $\lambda = (12, 7, 2) \qquad \mu = (3, 3, 2, 2, 1)$

$\pi = 1,2 \mid 1,2 \mid 1,2 \mid 1,1,1 \mid 1,1,1 \mid 1,2,2,3 \mid 1,2,2,3 \mid 1$

$$T = \begin{array}{|c|c|c|} \hline 1,2 & 1,2 & 1,2 \\ \hline & & \\ \hline 1,2,2,3 & 1,2,2,3 \\ \cline{1-2} 1,1,1 & 1,1,1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

$\downarrow$

$(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$

$\alpha_i^{(d)}$ = # labels d in the i^{th} row of T

$\alpha^{(1)} = (3, 0, 2, 6, 1) \Vdash_w 12 = \lambda_1$

$\alpha^{(2)} = (3, 0, 4, 0, 0) \Vdash_w 7 = \lambda_2$

$\alpha^{(3)} = (0, 0, 2, 0, 0) \Vdash_w 2 = \lambda_3$

$\mu_1 = 3 \quad \mu_2 = 3 \quad \mu_3 = 2 \quad \mu_4 = 2 \quad \mu_5 = 1$

This bijection proves thm. ■

Cor: $$h_\lambda = \tilde{h}_\lambda + \sum_{\gamma : |\gamma| < |\lambda|} a_{\lambda\gamma} \tilde{h}_\gamma$$

$a_{\lambda\gamma}$ = # $\pi \vdash \{\{1^{\lambda_1}, \dots, \ell^{\lambda_\ell}\}\}$ w/ $\gamma = m(\pi)$

$\{\tilde h_\lambda\}$ is a basis for Λ

<u>Mike</u>: $$\tilde h_\mu = \sum_{|\lambda|\le|\mu|} \underbrace{K_{(n-|\lambda|,\lambda),(n-|\mu|,\mu)}}_{\text{Kostka \#'s}} \overset{\tilde S_\mu}{}$$

Kostka #'s
= # SSYT of shape $(n-|\lambda|,\lambda)$
and content $(n-|\mu|,\mu)$

When $n \ge 2|\lambda|$, then $K_{(n-|\lambda|,\lambda),(n-|\mu|,\mu)}$ does not depend on n

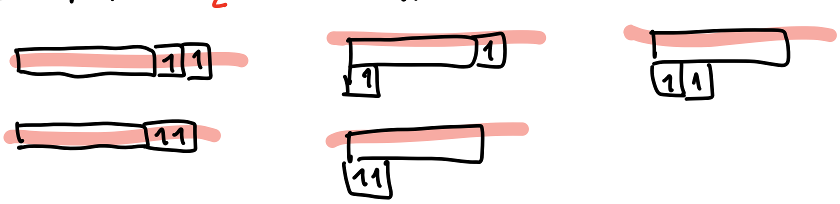
$\Rightarrow$ $\{\tilde S_\lambda\}$ is a basis

$\bar\lambda = (\lambda_2, \lambda_3, \dots)$

Thm: for any partition μ
$$h_\mu = \sum_{\lambda : |\bar\lambda| \le |\mu|} M_{\lambda\mu} \tilde S_{\bar\lambda}$$
$M_{\lambda\mu}$ = #SSYT of skew shape $\lambda/(\lambda_2)$
filled w/a multiset partition $\pi \vDash \{1^{\mu_1}, 2, \dots \ell^{\mu_\ell}\}$

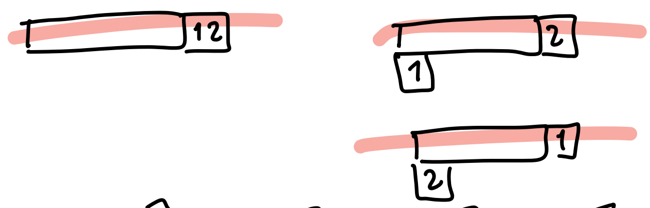
"lexicographic order" on multisets

<u>Examples</u>: h_2 $\{\{1,1\}\}$



$\Rightarrow$ $h_2 = 2S_\phi + 2\tilde S_{(1)} + \tilde S_{(2)}$

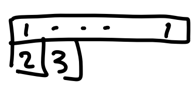
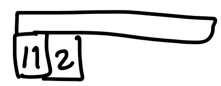
- $h_{1,1}$ $\{\{1,2\}\}$



$$h_{1,1} = 2\tilde{S}_\emptyset + 3\tilde{S}_{(1)} + \tilde{S}_{(2)} + \tilde{S}_{(1,1)}$$

Pf of Thm: $\tilde{h}_\lambda \longrightarrow \tilde{h}_\mu \longrightarrow \tilde{S}_\gamma$

Coeff. of $\tilde{S}_\gamma$ counts pairs: $(\underbrace{\Pi}_{\text{multiset partition}}, \underbrace{T}_{\text{Column strict tableaux}}) \xleftrightarrow{\text{Bij.}}$ Column strict tableaux filled with the multisets in Π.

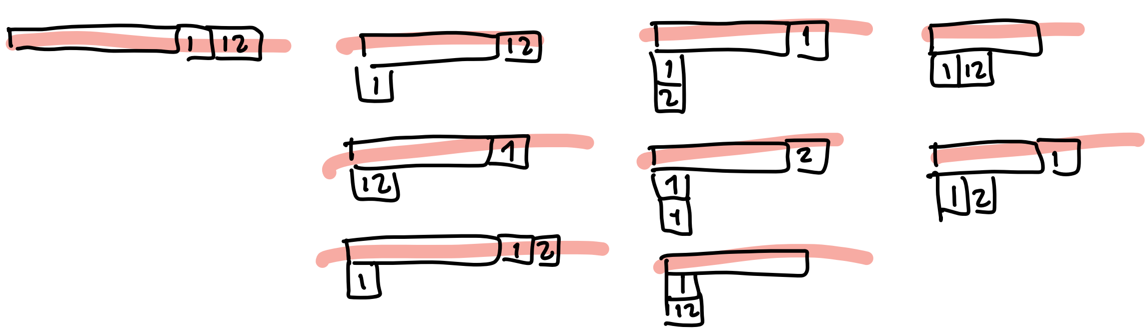
ex $(11|2,$  $) \longrightarrow$ 

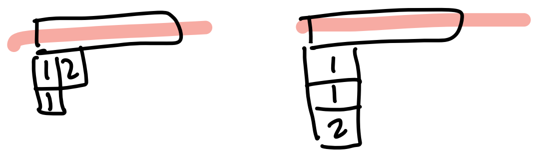
$e \longrightarrow \tilde{S}$

Thm: $$e_\mu = \sum_{\lambda : |\bar{\lambda}| \le |\mu|} N_{\lambda\mu} \tilde{S}_{\bar{\lambda}}$$

$N_{\lambda\mu}$ = # tableaux T of shape $\lambda/(\lambda_2)$
weakly inc in rows and in columns
filled with sets
AND: only even sets repeat in rows
" odd " columns

Example: $\lambda = (2,1)$ $\{\{1,1,2\}\}$





$$e_{2,1} = \tilde{s}_{\phi} + 3\,\tilde{s}_{(1)} + 3\,\tilde{s}_{1,1} + 2\,\tilde{s}_{(2)} + \tilde{s}_{(2,1)} + \tilde{s}_{(1,1,1)}$$

$\blacksquare$