Symmetric Group Characters as Symmetric Functions

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Lecture 6: Put it into Context and Wrap-up

Mike: Unified Combinatorial descriptions for the following:

1) $h_\mu \tilde{s}_\lambda$
2) $\tilde{h}_{\mu_1}\tilde{h}_{\mu_2}\cdots\tilde{h}_{\mu_\ell}\tilde{s}_\lambda$
3) $\tilde{s}_{\mu_1}\tilde{s}_{\mu_2}\cdots\tilde{s}_{\mu_\ell}\tilde{s}_\lambda$ ← contains Pieri Rule $\tilde{s}_r \tilde{s}_\lambda$
4) $\tilde{h}_\mu \tilde{s}_\lambda$

$$\begin{array}{ccccc}
\tilde{s}_{\mu_1}\cdots\tilde{s}_{\mu_\ell}\tilde{s}_\lambda & \le & \tilde{h}_{\mu_1}\cdots\tilde{h}_{\mu_\ell}\tilde{s}_\lambda & \le & h_\mu \tilde{s}_\lambda \\
\vee| & & \vee| & & \vee| \\
\tilde{s}_\mu \tilde{s}_\lambda & \le & \tilde{h}_\mu \tilde{s}_\lambda & \le & s_\mu \tilde{s}_\lambda
\end{array}$$

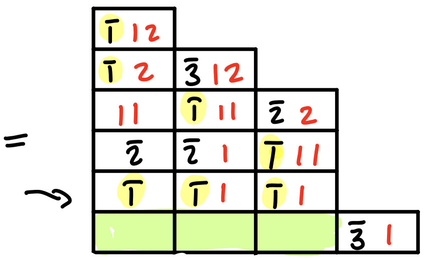
reduced Kronecker

Restriction Problem

Combinatorial objects: Multiset Tableaux.

letters: $\bar{1} < \bar{2} < \cdots < 1 < 2 < \cdots$

$T =$



$\bar{1}$ 12			
$\bar{1}$ 2	$\bar{3}$ 12		
11	$\bar{1}$ 11	$\bar{2}$ 2	
$\bar{2}$	$\bar{2}$ 1	$\bar{1}$ 11	
$\bar{1}$	$\bar{1}$ 1	$\bar{1}$ 1	
			$\bar{3}$ 1

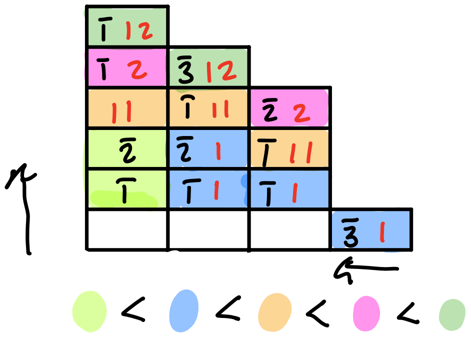
- Boxes are filled w/ multiset and at most one barred entry.
- Column strict wrt reverse lex order
- Shape of T: $\gamma/(3)$
  $(4,3,3,3,2,1)/(3)$

Content:

- Barred $\alpha = (7,3,2)$
- unbarred $\beta = (12, 4)$

$MT_\gamma(\underset{\text{Barred content}}{\lambda}, \underset{\text{unbarred content}}{\mu})$ = Set of all these tableaus of Shape $\gamma = \text{shape}(\underset{\text{remove first row}}{\overline{T}})$

Lattice Condition: (more i's than $(i+1)$'s)



$\phi \le \{\{1\}\} \le \{\{1,1\}\} \le \{\{2\}\} \le \{\{1,2\}\}$

$\bar1\bar2\ \bar3\bar1\bar1\bar2\ \bar1\bar1\ \bar2\bar1\ \bar3\bar1$ ✓

Check if lattice

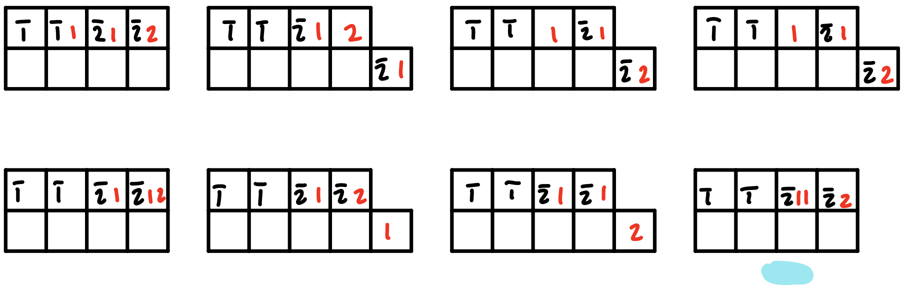
Combinatorial Interpretations

	Coeff of $\tilde{S}_\gamma$
$h_\mu \tilde{S}_\lambda$	# $T \in MT_\gamma(\lambda,\mu)$ s.t. T is lattice
$\tilde{h}_{\mu_1}\cdots\tilde{h}_{\mu_\ell}\tilde{S}_\lambda$	# $T \in MT_\gamma(\lambda,\mu)$ s.t. T is lattice and entries are sets (no repetitions in boxes)
$\tilde{h}_\mu \tilde{S}_\lambda$	# $T \in MT_\gamma(\lambda,\mu)$ s.t. T is lattice the sets allowed are $\{\bar{i}, j\}$, $\{\bar{i}\}$, $\{j\}$
$\tilde{S}_{\mu_1}\cdots\tilde{S}_{\mu_\ell}\tilde{S}_\lambda$	# $T \in MT_\gamma(\lambda,\mu)$ s.t T is lattice and has set entries & no sets of size are allowed in first row of $(n-\lvert\gamma\rvert, \sigma)$.

Example: $\mu = (2,1)$ and $\lambda = (2,2)$ $1,1,2,\overline{1},\overline{1},\overline{2},\overline{2}$

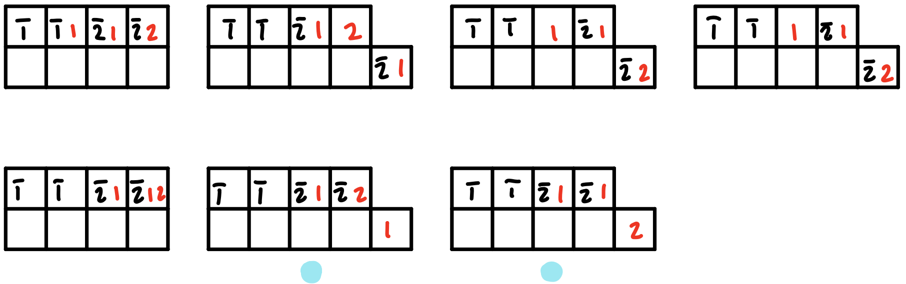
Coefficient of $\tilde{S}_{(4)}$

$h_{21}\,\tilde{S}_{22}$



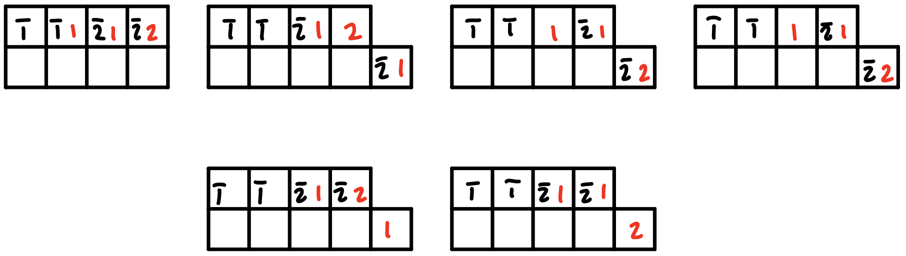
⑧

$\tilde{h}_2\,\tilde{h}_1\,\tilde{S}_{2,2}$



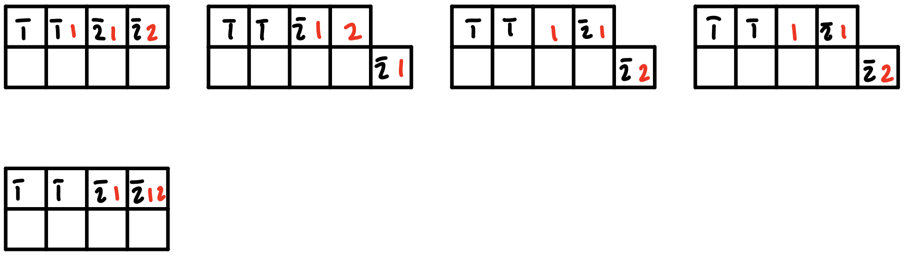
⑦

$\tilde{h}_{2,1}\,\tilde{S}_{2,2}$



⑥

$\tilde{S}_2\,\tilde{S}_1\,\tilde{S}_{2,2}$



⑤

Main Theorem:

There exists a non-homogeneous basis $\{\tilde{S}_\lambda\}$ of Symm. fncs. characterized by any of the following:

(1) $n \geq |\lambda| + \lambda_1$

$$\tilde{S}_\lambda\underbrace{(x_1, \dots, x_n)}_{\text{eigenvalues of perm. matrix of cycle type } \mu} = \chi_{S_n}^{(n-|\lambda|, \lambda)}(\mu)$$

② $\mathbb{V}^\lambda$ is a poly rep of Gl_n (irred)

then

if $\mathbb{V}^\lambda \big\downarrow^{Gl_n}_{S_n} \cong \bigoplus_\mu \left(S^{(n-|\mu|,\mu)}\right)^{\oplus r_{\lambda\mu}}$

$r_{\lambda\mu}$ restriction coeffs.

then $s_\lambda = \sum_\mu r_{\lambda\mu} \tilde{S}_\lambda$

③ $S_{1^r} = \tilde{S}_{1^r} + \tilde{S}_{1^{r-1}}$ $\left(\Leftrightarrow \tilde{S}_{1^r} = \sum_{i=0}^{r} (-1)^i e_{r-i}\right)$

$r \geq 0$

$$\tilde{S}_\lambda \tilde{S}_\mu = \sum_\nu \bar{g}_{\lambda\mu}^\nu \tilde{S}_\nu$$

S_{1^r} is the character of $\Lambda^r(\mathbb{C}^n)$

$$\Lambda^r(\mathbb{C}) \big\downarrow^{Gl_n}_{S_n} \cong S^{(n-r,1^r)} \oplus S^{(n+1-r,1^{r-1})}$$

Symm. fncs as characters of GL_n

Rep of GL_n	Character at element w/ eigenvalues $x_1,\dots,x_n$
$\mathbb{V}^{\lambda}$	$s_\lambda(x_1,\dots,x_n)$
$Sym^{\lambda_1}(\mathbb{C}^n)\otimes\cdots\otimes Sym^{\lambda_\ell}(\mathbb{C}^n)$	$h_\lambda(x_1,\dots,x_n)$
$\Lambda^{\lambda_1}(\mathbb{C}^n)\otimes\cdots\otimes\Lambda^{\lambda_\ell}(\mathbb{C}^n)$	$e_\lambda(x_1,\dots,x_n)$
$\mathbb{V}^{\lambda}\otimes\mathbb{V}^{\mu}$	$s_\lambda s_\mu = \sum c^{\nu}_{\lambda,\mu} s_\nu$
$V^{\otimes k}$ as a rep of GL_n	$h_{1^k} = \sum \underset{\uparrow \#SYT}{f^{\lambda}} s_\nu$
$V^{\otimes k}$ as a rep of $GL_n \times S_k$	$\underset{\uparrow \text{cycle type of an element of } S_k}{p_\mu}(x_1,\dots,x_n)$

Sym. fncs as characters of S_n

Rep S_n	Character at element of $\overset{\text{cycle type}\downarrow}{\mu}$
$\mathcal{S}^{\lambda}$	$\tilde{s}_{\bar{\lambda}}(\Xi_\mu)$
$M^{\lambda}\cong \mathbb{1}\uparrow_{S_{\lambda_1}\times\cdots\times S_{\lambda_\ell}}^{S_n}$	$\tilde{h}_{\bar{\lambda}}(\Xi_\mu)$
$\mathcal{S}^{(n-\lambda_1,\lambda_1)}\otimes\cdots\otimes\mathcal{S}^{(n-\lambda_\ell,\lambda_\ell)}$	$\to \mathbb{D}$ $\tilde{s}_{\lambda_1}\cdots\tilde{s}_{\lambda_\ell}$ mult. $\updownarrow$
$M^{(n-\lambda_1,\lambda_1)}\otimes\cdots\otimes M^{(n-\lambda_\ell,\lambda_\ell)}$	$\tilde{h}_{\lambda_1}\cdots\tilde{h}_{\lambda_\ell}$ mult.
$\mathcal{S}^{(n-\lvert\lambda\rvert,\lambda)}\otimes\mathcal{S}^{(n-\lvert\mu\rvert,\mu)}$	$\tilde{s}_\lambda\tilde{s}_\mu = \sum \bar{g}^{\nu}_{\lambda\mu}\tilde{s}_\nu$

$(\mathbb{C}^n)^{\otimes k}$

as a rep of $S_n \times P_k(n)$

partition algebra

$$P_\gamma(\Xi_\mu) = \sum \chi^{\lambda}_{P_n(n)}(d_\mu)\,\tilde{s}_\lambda$$

"conjugacy class" rep of partition algebra

$0 \le |\gamma| \le k$

Murnaghan-Nakayama Rule:

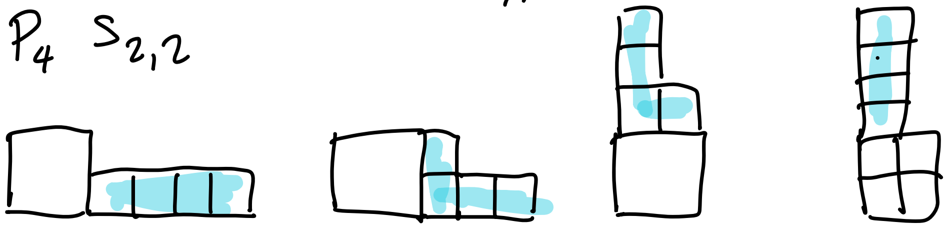
Classical Case:

$$P_\mu = \sum_\lambda \chi^\lambda(\mu)\, s_\lambda$$

Thm: $$P_k s_\mu = \sum_{\mu \subseteq \lambda} (-1)^{ht(\lambda/\mu)-1} s_\lambda$$

λ/μ is a rim hook

ex: $P_4\ s_{2,2}$



Thm: (Halverson, O-Z)

$$P_k \tilde{s}_\lambda = \sum_{\substack{\nu \\ |\nu| \le k+|\lambda|}} \left(\sum_{d|k} \sum_{\alpha} (-1)^{ht(\lambda//\alpha)-2+ht(\nu//\alpha)} \right) \tilde{s}_\nu$$

add long first row

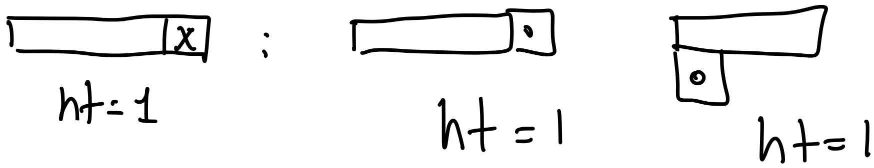
α: partitions s.t. $\lambda//\alpha$ and $\nu//\alpha$ are rim hooks

Example: $p_{2,1} = p_2 p_1$

$p_1 \tilde{s}_\phi$

$d=1$

$K=1$

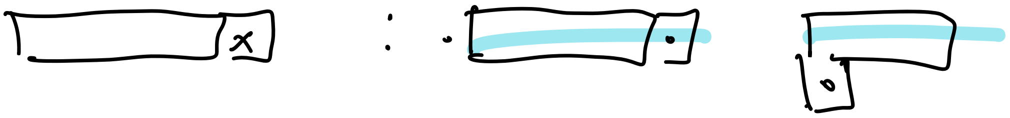


$$\Rightarrow \quad p_1 = \tilde{s}_\phi + \tilde{s}_1$$

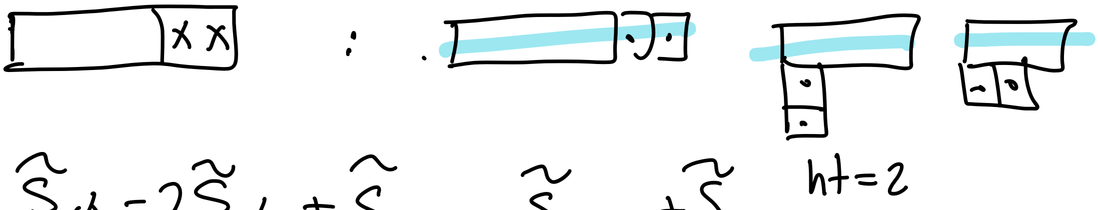
$$p_2(\tilde{s}_\phi + \tilde{s}_1) = \underline{p_2 \tilde{s}_\phi} + p_2 \tilde{s}_1$$

$K=2$ $\quad \underline{p_2 \tilde{s}_\phi}$

$d=1$



$d=2$



$$p_2 \tilde{s}_\phi = 2\tilde{s}_\phi + \tilde{s}_{(1)} - \tilde{s}_{(111)} + \tilde{s}_{(2)}$$

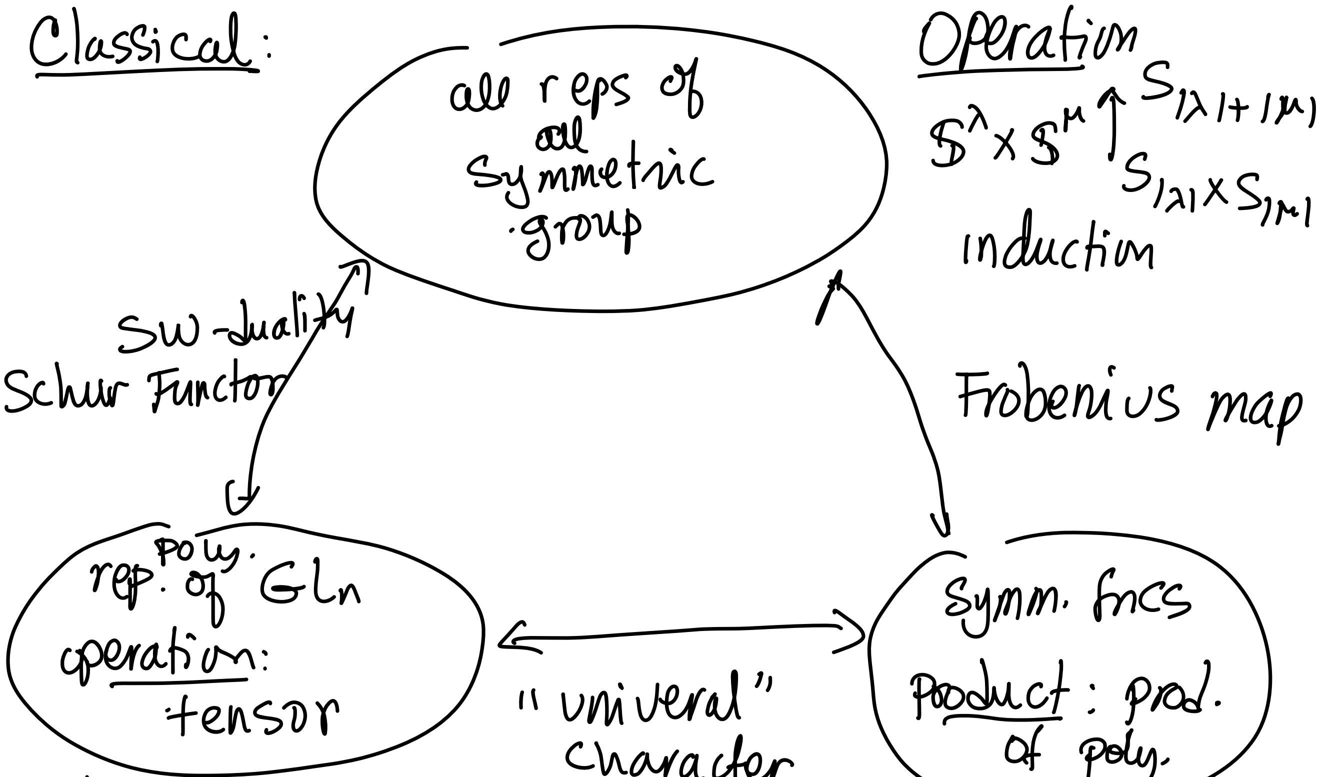
$$p_2 \tilde{s}_1 = \tilde{s}_\phi + 3\tilde{s}_{(1)} + \tilde{s}_{(2)} + \tilde{s}_{(1,1)} + \tilde{s}_{(3)}$$
$$- \tilde{s}_{(1,1,1,1)}$$

$$p_{(2,1)} = 3\tilde{s}_\phi + 4\tilde{s}_{(1)} + 2\tilde{s}_{(2)} + \tilde{s}_{(3)} - \tilde{s}_{(1,1,1,1)}$$

Character table of $P_3(n)$

	ϕ	(1)	(2)	(1^2)	(3)	$(2,1)$	(1^3)
ϕ	n^3	n^2	$2n$	$2n$	2	3	5
(1)	0	n^2	n	$3n$	1	4	10
(2)	0	0	n	n	0	2	6
$(1,1)$	0	0	$-n$	n	0	0	6
(3)	0	0	0	0	1	1	1
$(2,1)$	0	0	0	0	-1	0	2
(1^3)	0	0	0	0	1	-1	1

Classical:



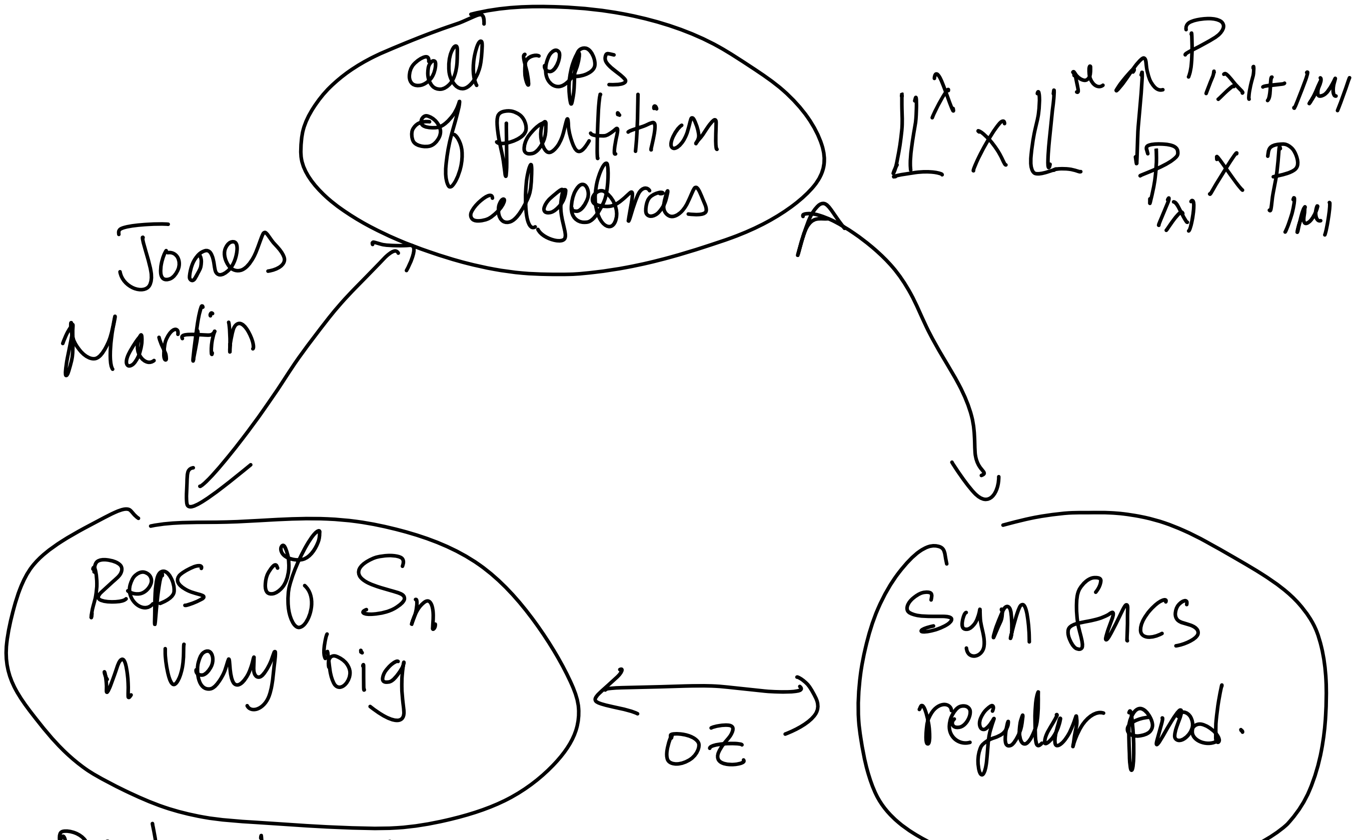
Combinatorics:

- tableaux filled w/ sets size 1
- RSK

$$\underset{i_1\ i_2\ \cdots\ i_k}{v_{i_1}\otimes\cdots\otimes v_{i_k}} \to (\underset{\uparrow\text{SSYT}}{P}, \underset{\nwarrow\text{SYT}}{Q})$$

- Littlewood-Richardson Rule.

Restricted Picture



$\mathbb{L}^{\lambda} \times \mathbb{L}^{\mu} \uparrow_{P_{|\lambda|} \times P_{|\mu|}}^{P_{|\lambda|+|\mu|}}$

Product: tensor

Combinatorics:

1) Tableaux are filled w/ multisets
2) RSK (COSSZ)
3) Pieri-Rule

To be continoed....

$$e_\mu = \sum \underset{\sim}{N_{\lambda\mu}} \tilde{S}_\lambda$$

↑ set filled tableaux

- Row strict on odd sets
- Column strict on even sets

$$e_\mu = \sum \underset{\sim}{\tilde{K}_{\lambda\mu}} S_\lambda$$

↖ filled w/ #'s (size 1 sets)

- Row strict.

$e_{2,2}$

1,1, 2,2

$$\begin{array}{l} 1 \\ 1 \\ 2 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 1 & \\ 2 & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 1 & 2 \end{array}$$