Lorentzian polynomials

Petter Brändén, KTH Royal Institute of Technology

Based on joint work with
June Huh and Jonathan Leake

Outline

1). Introduction, Stable polynomials,
M-convex sets
2). Lorentzian polynomials on cones.
3). characterization of Lorentzian polynomials.
4). Mason's conjecture.
5). Heron-Rota-Welsh conjecture on
the characteristic polynomial of a matroid.
6). ?

Stable polynomials

$a_0, a_1, a_2, \dots, a_n$ pos. numbers
We are interested in the shape.

unimodality: $a_0 \le a_1 \le \dots \le a_m \ge a_{m+1} \ge \dots$

$\Uparrow$ $\qquad a_k \ge \min\{a_{k-1}, a_{k+1}\}$

log-concavity: $a_k^2 \ge a_{k-1} a_{k+1}$

$\Uparrow$

Newton's inequalities $\dfrac{a_k^2}{\binom{n}{k}^2} \ge \dfrac{a_{k-1}}{\binom{n}{k-1}} \cdot \dfrac{a_{k+1}}{\binom{n}{k+1}}$

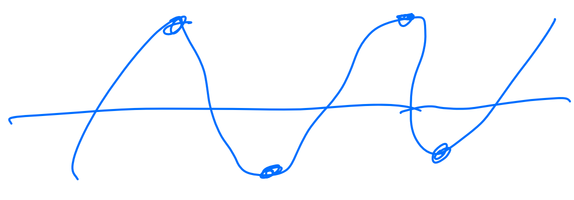
$\Uparrow$ *

Real-rootedness of $a_0 + a_1 x + \dots + a_n x^n$

Proof of *: $f = \sum_{k=0}^{n} \underbrace{\binom{n}{k} b_k}_{= a_k} x^k$

Suppose f is real-rooted

(a) f' is real-rooted



$$f' = n \sum_{k=0}^{n-1} b_{k+1} \binom{n-1}{k} x^k$$

(b) $x^n f\left(\frac{1}{x}\right) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} x^k$ is real-rooted.

Using (a) and (b) deduce that

$$\binom{2}{0} b_{k-1} + \binom{2}{1} b_k x + \binom{2}{2} b_{k+1} x^2$$

is real-rooted. Taking discriminant,

$$b_k^2 \geq b_{k-1} b_{k+1}$$

Examples: Eulerian polynomials,
Matching polynomials, rook pol., orthogonal pol., characteristic pol. of symmetric matrices.

- stability is a multivariate analog of real-rootedness

Definition: A polynomial $f = \mathbb{C}[x_1, \dots, x_n]$ is stable if $f(x_1, \dots, x_n) \neq 0$ whenever all variables have positive imaginary parts. If in addition $f \in \mathbb{R}[x_1, \dots, x_n]$ we call f real-stable.

Example: • $2x_1 + 3x_3 + x_n - 3$ is real-stable.

• $x_1 x_2 - 1 = x_1(x_2 - x_1^{-1})$ also

• If $f \in \mathbb{R}[x]$, then f is stable iff it is real-rooted (non-real zeros come in conjugate pairs)

• Stability is a closed property

$f_n \to f$, $\quad (\deg f_n \le C)$

then f is stable, if all f_n are stable.

• Determinantal polynomials:
$f = \det(x_1 A_1 + \dots + x_n A_n + A_0)$,

where $A_1, A_2, \dots, A_n$ are PSD

hermitian matrices, and A_0 is hermitian.

We may assume that $A_1, \dots, A_n$ are PD.

Let $x_j = a_j + i b_j$, $a_j \in \mathbb{R}$, $b_j > 0$.

$$f(x_1, \dots, x_n) = \det\left(\sum_{j=1}^{n} a_j A_j + A_0 + i\left(\sum_{j=1}^{n} b_j A_j\right)\right)$$

$$= \det(A + iB), \quad B \text{ is PD}$$

$$(B = PP^*, \quad P \in GL_n)$$

$$= \det(B) \cdot \det(\underbrace{P^{-1}A(P^{-1})^*}_{\text{hermitian}} + iI) \neq 0,$$

since $-i$ is not an eigenvalue of a hermitian matrix. $\square$

Helton–Vinnikov theorem:

Any real-stable polynomial of degree d, $f(x,y)$, may be written

$$f(x,y) = \det(xA + yB + C)$$

where A, B, C are real-symmetric $d \times d$ matrices and A, B are PSD.

- Fails for more than 2 variables.

Notice that $f \in \mathbb{R}[x_1, \dots, x_n]$ is stable iff for all $\alpha \in \mathbb{R}^n_{>0}$, $\beta \in \mathbb{R}^n$

$$f(\alpha t - \beta) \text{ is real-rooted.}$$

Closure properties of stable pol.

- If f, g are stable, then so is fg.

- If $v_1, \dots, v_m \in \mathbb{R}^n_{\geq 0}$ and $v_0 \in \mathbb{R}^n$ and $f \in \mathbb{R}[x_1, \dots, x_n]$ is stable, then

  so is $g(t_1, \dots, t_m) = f(v_0 + t_1 v_1 + \dots + t_m v_m)$
  (unless $g \equiv 0$). $\left(= f\left(v_0 + A \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}\right) \right)$

  non-negative entries.

- If f is stable and of degree at

most d in x_1, then

$$x_1^d f\left(-\frac{1}{x_1}, x_2, x_3, \ldots, x_n\right)$$

is stable. maps the upper-half-plane to itself.

- If f stable and $v \in \mathbb{R}^n_{\geq 0}$, then

$$D_v f = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i} \quad \text{is stable. (or } \equiv 0\text{)}$$

Proof:

Write $f(x) = \sum_{k=0}^d a_k(x_2, \ldots, x_n) x_1^k$

$a_d \neq 0$

$$a_d(x_2, \ldots, x_n) = \lim_{t \to \infty} t^{-d} f(ti, x_2, \ldots, x_n) i^d$$

is stable.

Prove that $\frac{\partial f}{\partial x_1}$ is stable.

Suffices to prove that for $b_2, b_3, \ldots, b_n$ in the upper half plane, then $g'(x)$, where $g(x) = f(x, b_2, b_3, \ldots, b_n)$, is stable.

$g(x)$ is stable, so

$$g(x) = c \cdot \prod_{i=1}^{n} (x - \zeta_j) , \quad \operatorname{Im}(\zeta_j) \le 0$$

$$\frac{g'(x)}{g(x)} = \sum_{i=1}^{n} \frac{1}{x - \zeta_j}$$

Let $\operatorname{Im}(x) > 0$

$$\operatorname{Im}\left[\frac{g'(x)}{g(x)}\right] = \sum_{i=1}^{n} \operatorname{Im}\left[\frac{1}{x - \zeta_j}\right] < 0$$

so $g'(x) \neq 0$.

Hence $\frac{\partial f}{\partial x_1}$ is stable.

If $v \in \mathbb{R}^n_{\ge 0}$, then

$f(x + x_0 v)$ is stable.

$= h(x_0, x_1, \dots, x_n)$

$$\left.\frac{\partial h}{\partial x_0}\right|_{x_0 = 0} = D_v f \quad \text{is stable (or } \equiv 0).$$