Recap.

- $f \in \mathbb{R}[x_1, \dots, x_n]$ is (real-) stable if

$$f(x_1, \dots, x_n) \neq 0 \quad \text{whenever } \operatorname{Im}(x_j) > 0 \ \forall j$$

Equivalently if $t \to f(te - x)$ is real-rooted for all $e \in \mathbb{R}^n_{>0}$, $x \in \mathbb{R}^n$

- f stable $\Longrightarrow$ $D_v f = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$ stable for all $v \in \mathbb{R}^n_{\geq 0}$.

- If A_0 is hermitian, and $A_1, \dots, A_n$ are PSD, then

$$f(x) = \det(A_0 + x_1 A_1 + \dots + x_n A_n)$$

is stable.

Example Multivariate Eulerian polynomials

For $\sigma = \sigma_1 \sigma_2 \dots \sigma_n \in \mathfrak{S}_n$, let

$$DT(\sigma) = \{\sigma_i : \sigma_i > \sigma_{i+1}\}$$
$$AT(\sigma) = \{\sigma_i : \sigma_{i-1} < \sigma_i\}$$
, where $\sigma_0 = \sigma_{n+1} = 0$

$$A_n(x,y) = \sum_{\sigma \in \mathfrak{S}_n} x^{DT(\sigma)} y^{AT(\sigma)}, \quad x^S = \prod_{i \in S} x_i$$

A_1 $\quad$ 010

$\shortparallel$

$x_1 y_1$

Insert $n+1$ in a permutation $\sigma \in S_n$

$$\sigma_i \underset{\underset{n+1}{\uparrow}}{>} \sigma_{i+1}. \quad \text{Effect} \quad x^{DT(\sigma)} y^{AT(\sigma)}$$
$$\downarrow$$
$$x_{n+1} y_{n+1} \frac{\partial}{\partial x_{\sigma_i}} x^{DT(\sigma)} y^{AT(\sigma)}$$

Recursion

$$A_{n+1} = x_{n+1} y_{n+1} \left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i} \right) \right) A_n$$

preserves stability $\qquad$ preserves stability

By induction A_n is stable.

Example Plücker-polynomial

$A = [v_1, v_2, \dots, v_n] \in \mathbb{C}^{r \times n}$, rank $= r$

$f = \det(x v_1 v_1^* + x_2 v_2 v^* + \dots + x_n v_n v_n^*)$

is stable

$$f = \det\left(A \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} A^* \right)$$

Cauchy-Binet

$$= \sum_{\substack{S \subseteq [n] \\ |S| = r}} \det(A(S)) \overline{\det(A(S))} \prod_{i \in S} x_i$$

$\uparrow$ $r \times r$ submatrix lying in rows S

support of $f = \{ S : \text{coeff. of } S \text{ is nonzero} \}$

$=$ set of bases of the linear matroid defined by $v_1, \dots, v_n$.

Characterization of homogeneous stable quadratics

Sylvester's law of inertia: Let A be a real symmetric $n \times n$ matrix. Then $P^T A P$ has the same signature as A for all $P \in GL_n(\mathbb{R})$.

Cauchy interlacing: The eigenvalues of any principal submatrix of a real symmetric matrix A interlaces the eigenvalues of A.

$$\lambda_n \le \mu_{n-1} \le \lambda_{n-1} \le \mu_{n-2} \le \cdots \le \lambda_2 \le \mu_1 \le \lambda_1$$

eigenvalues of A — eigenvalues of submatrix

Any hom. quadratic may be written

$$f = \sum_{i,j} a_{ij} x_i x_j = x^T A x$$

$$A = \frac{1}{2} \left(\partial_i \partial_j f \right)_{i,j=1}^{n} = \frac{1}{2} \nabla f$$

Prop. Let $f = x^T A x$, where A is a symmetric $n \times n$ matrix with nonnegative coeff. TFAE

(1). A has exactly one positive eigenvalue.

(2). f is stable (and $\not\equiv 0$)

(3). $(e^TAx)^2 \geq (e^TAe)(x^TAx)$
for all $e \in \mathbb{R}^n_{>0}$ and $x \in \mathbb{R}^n$

(4). — // —
for some $e \in \mathbb{R}^n$ s.t. $e^TAe > 0$
and all $x \in \mathbb{R}^n$

<u>Proof</u>: $f(te - x) = (te-x)^TA(te-x)$, $e \in \mathbb{R}^n_{>0}$

$$= t^2(e^TAe) - 2t(e^TAx) + (x^TAx)$$

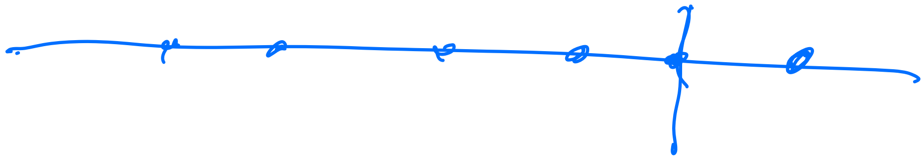
stable $\Longleftrightarrow$ $(e^TAx)^2 \geq (e^TAe)(x^TAx)$

which proves $(2) \Leftrightarrow (3)$.

$(1) \Rightarrow (2)$: Take $e \in \mathbb{R}^n_{>0}$ and $x \in \mathbb{R}^n$.
If $e \| x$, then (3) is obvious.
Assume $e \not\| x$. Extend to a basis
$P = [e, x, v_3, \dots, v_n] \in GL_n(\mathbb{R})$
Sylvester implies

$$P^TAP = \begin{bmatrix} e^TAe & e^TAx & \cdots \\ e^TAx & x^TAx & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

has exactly one pos. eigenvalue



Cauchy interlacing says that

$$B = \begin{bmatrix} e^TAe & e^TAx \\ e^TAx & x^TAx \end{bmatrix}$$ has at most one pos. eigenvalue.

Because $e^TAe > 0$, $e_1^T B e_1 > 0$, (standard basis vector)

so B is not NSD; so it has exactly one pos. eigenvalue.

$\therefore \quad \det(B) \le 0$

$\quad\quad = (e^TAe)(x^TAx) - (e^TAx)^2$

$(3) \Rightarrow (4)$ trivially

$(4) \Rightarrow (1)$. Assume (4). Then $e^TAe > 0$

$$(e^TAx)^2 \ge (e^TAe)\cdot(x^TAx), \quad \forall x \in \mathbb{R}^n$$

Let

$$H = \{ x \in \mathbb{R}^n : e^TAx = 0 \}$$ hyperplane

Then A (restricted to H) is NSD.

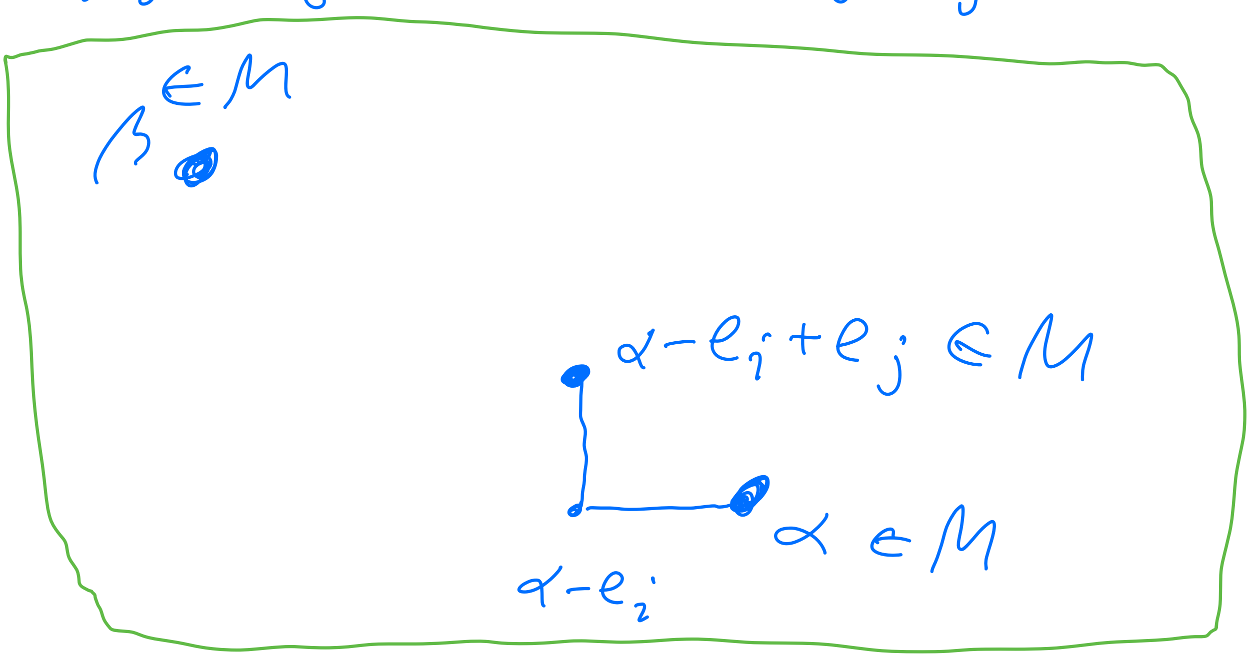
By Cauchy and Sylvester we see that A has at most one positive eigenvalue. Since $e^T A e > 0$ it has a positive eigenvalue. $\square$

<u>M-convex sets</u> (aka polymatroids, integer points of generalized permutahedra)

<u>Def</u>. M finite subset $\mathbb{N}^n$. M is called M-convex if

(EA) $\alpha, \beta \in M$, $\alpha_i > \beta_i \implies$

$\exists j \;\; \beta_j > \alpha_j$ and $\alpha - e_i + e_j \in M$



<u>Example</u>: If $M \subseteq \{0,1\}^n$, then M

is M-convex iff M is the set of bases of a matroid.

<u>Def</u>. If $f = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) x^\alpha$ $\quad \left(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right)$

The support of f is

$$\mathrm{supp}(f) = \{ \alpha \in \mathbb{N}^n : a(\alpha) \neq 0 \}$$

$i \in [n]$: $\quad \partial_i M = \{ \alpha - e_i : \alpha \in M, \ \alpha_i > 0 \}$
(contraction)

$$\tau M = \bigcup_{i \in [n]} \partial_i M \qquad \text{(truncation)}$$

$a(\alpha) \geq 0$: $\quad \mathrm{supp}(\partial_i f) = \partial_i \, \mathrm{supp}(f)$
$v \in \mathbb{R}^n_{>0}$: $\quad \mathrm{supp}(D_v f) = \tau \, \mathrm{supp}(f)$.

- If M is M-convex, then so are $\partial_i M$ and τM.

<u>Lemma</u> Suppose $M \subseteq \mathbb{N}^n$, $\sum_{i=1}^n \alpha_i > 2$, for some $\alpha \in M$.

If $\partial_i M$ is M-convex for all i
τM is M-convex,

then M is M-convex,