## Recap

- $M \subseteq \mathbb{N}^n$ is M-convex if

(EA) $\quad \alpha, \beta \in M,\ \alpha_i > \beta_i \Rightarrow \exists\, j\ \beta_j > \alpha_j,\ \alpha - e_i + e_j \in M$

Contraction: $\partial_i M = \{\alpha - e_i : \alpha \in M,\ \alpha_i > 0\}$

Truncation: $\tau M = \bigcup_{i=1}^{n} \partial_i M$

- If $M$ is M-convex, then so are $\partial_i M$ and $\tau M$

  converse:

Lemma R. Suppose $M \subseteq \{\alpha \in \mathbb{N}^n : \sum_{i=1}^{n} \alpha_i = r\}$, $r \geq 3$. If $\tau M$ and $\partial_i M$ are M-convex for all $i$, then $M$ is M-convex.

$f = \sum_\alpha a_\alpha x^\alpha$
$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0\}$

Remark: If $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is homogeneous, then $\mathrm{supp}(\partial_i f) = \partial_i \,\mathrm{supp}(f)$ and $\mathrm{supp}(D_v f) = \tau\, \mathrm{supp}(f)$ for all $v \in \mathbb{R}^n_{>0}$.

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Theorem (Choe–Oxley–Sokal–Wagner, 2004). If $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is homogeneous and stable, then $\mathrm{supp}(f)$ is M-convex.

Proof. Since stable polynomials are closed under $\partial_i$ and $D_v$, the supports of such polynomials are closed under $\partial_i$ and $\tau$. It remains to consider $d = 2$.

The support of quadratic stable pol., are M-convex:

<u>Proof</u>. Previous lecture we may assume $f(x) = x^T A x$, $A$ is symmetric

$A = (a_{ij})_{i,j=1}^{n}$, $a_{ij} \geq 0$

$$(\ast) \qquad (x^T A y)^2 \geq (x^T A x)(y^T A y), \quad x \in \mathbb{R}^n_{\geq 0}, \; y \in \mathbb{R}^n$$

Proof by contradiction. Assume

$$\alpha = e_i + e_j \in M \quad \text{violate (EA).}$$
$$\beta = e_k + e_l \in M$$

Then $\{i,j\} \cap \{k,l\} = \emptyset$

$\neg(EA)$: $\alpha - e_i + e_k = e_j + e_k \notin M$

$\alpha - e_i + e_l = e_j + e_l \notin M$

$$x = e_i + t e_j, \quad t > 0$$
$$y = e_k + e_l$$

$$x^T A y = a_{ik} + a_{il} \qquad (a_{jk} = 0)$$
$$x^T A x = a_{ii} + t^2 a_{jj} + 2t \underset{\geq 0}{a_{ij}}$$
$$y^T A y = a_{kk} + a_{ll} + 2 \underset{\geq 0}{a_{kl}}$$

if $t \to \infty$ then the RHS of $(\ast)$ dominates. $\square$

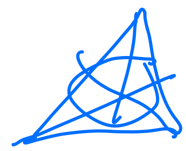
<u>Question</u>: Which matroids are supports of stable polynomials?

<u>Yes</u>: • Representable over $\mathbb{C}$.

• Vamós matroid (Wagner, Wei)

(Used to construct counterexamples to generalized Lax conjecture)

<u>No</u>: (B., Gonzaléz D'León)

• No projective geometry 

• No binary-non-regular.

# C-Lorentzian polynomials

Minkowski volume polynomials:

$K_1, \dots, K_n$ convex bodies in $\mathbb{R}^d$:

$$\mathrm{vol}(\underbrace{x_1K_1 + \dots + x_nK_n}_{\text{Minkowski sum}}) = \sum_{i_1,\dots,i_d} \underbrace{V(K_{i_1}, K_{i_2}, \dots, K_{i_d})}_{\text{Mixed volumes } \geq 0} x_{i_1}x_{i_2}\cdots x_{i_d}$$

Alexandrov-Fenchel inequalities:

$$V(K_1, K_2, K_3, \dots, K_d)^2 \geq V(K_1, K_1, K_3, \dots, K_d) \cdot V(K_2, K_2, K_3, \dots, K_d)$$

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Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be homogeneous of deg. $d$ and $v_1, \dots, v_m \in \mathbb{R}^n$. Then

$$f(t_1v_1 + \dots + t_mv_m) = \frac{1}{d!} \sum_{i_1,\dots,i_d} \underbrace{(D_{v_{i_1}} D_{v_{i_2}} \cdots D_{v_{i_d}} f)}_{\text{mixed derivatives}} \cdot t_{i_1} t_{i_2} \cdots t_{i_d}$$

- Let $C \subseteq \mathbb{R}^n$ be a convex open cone.

**Def:** $f$ is $C$-Lorentzian if for all $v_1, v_2, \dots, v_d \in C$,

(P) $\quad D_{v_1} D_{v_2} \cdots D_{v_d} f > 0$

(AF) $\quad (D_{v_1} \cdots D_{v_d} f)^2 \geq (D_{v_1} D_{v_1} D_{v_3} \cdots D_{v_d} f) \cdot (D_{v_2} D_{v_2} D_{v_3} \cdots D_{v_d} f).$

we can replace (AF) with an equivalent condition:

(L) The Hessian of $D_{v_1} D_{v_2} \cdots D_{v_{d-2}} f$ has exactly one pos. eigenvalue.

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- If $C = \mathbb{R}^n_{>0}$, then we say that $f$ is Lorentzian.

In this case
$(L) \Leftrightarrow D_{v_1} \cdots D_{v_{d-2}} f$ is stable.

__Remark__: Since stable pol. are closed under $D_v f$, $v \in \mathbb{R}^n_{>0}$, and if $f \not\equiv 0$ has nonnegative coefficients, then

(P) $D_{v_1} \cdots D_{v_d} f > 0$

(L) $D_{v_1} \cdots D_{v_{d-2}} f$ is stable.

it follows that homogenous stable polynomials are Lorentzian.

## Examples

- Stable homogeneous polynomials are Lorentzian.
- The polynomial $\det(X)$ is $C$-Lorentzian
  $C = \{\text{positive definite symmetric matrices}\}$
- Minkowski volume polynomials are Lorentzian.
- Normalized Schur and Schubert polynomials are Lorentzian
- Various Matroid polynomials are Lorentzian.

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- $C$-Lorentzian is a closed property.
- If $v_1, v_2 \in \overline{C}$, then

$$f(sv_1 + tv_2) = \frac{1}{d!} \sum_{k=0}^{d} \binom{d}{k} \underbrace{\left(D_{v_1}^{k} D_{v_2}^{d-k} f\right)}_{= a_k} s^k t^{d-k}$$

$$(AF) \Rightarrow \quad a_k^2 \geq a_{k-1} a_{k+1}, \quad 0 < k < d.$$

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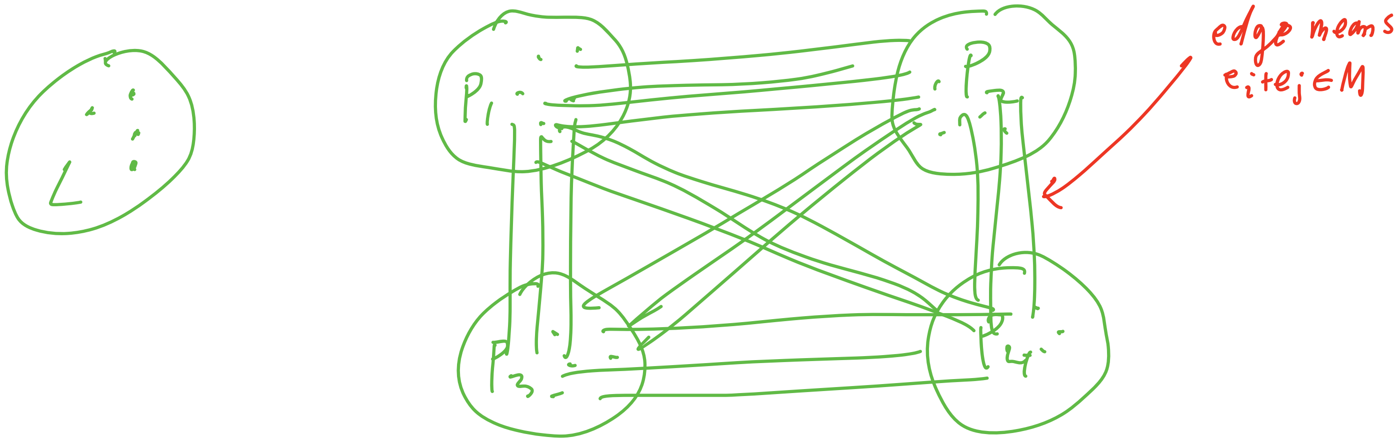
Non-example $x^3 + y^3$. (See later)

- How does a rank 2 M-convex set look like?
- If $M \subseteq \{0,1\}^n$. Let $L \subseteq E$ be the loops, and let $P_1, \dots, P_m$ be the parallel classes.



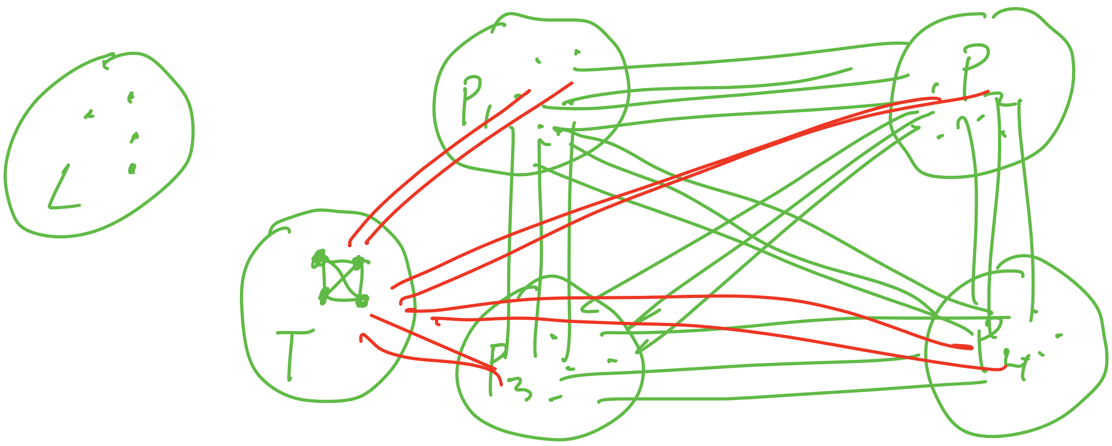

complete graph.

- If $M \subseteq \{0,1,2\}^n$.

Suppose $\underbrace{2e_k}_{\beta} \in M$ and $\underbrace{e_i + e_j}_{\alpha} \in M$

Then $\alpha - e_i + e_k = e_j + e_k \in M$ and
$\alpha - e_j + e_k = e_i + e_k \in M$

Let $T = \{k : 2e_k \in M\}$



Perron – Frobenius:

We say that a matrix $A = (a_{ij})_{i,j=1}^{n}$ is regular if:

(P) All off-diagonal entries are nonnegative

(C) For every distinct $i, j \in [n]$, there is a sequence $i = i_0, i_1, \dots, i_k = j$ s.t.

$$a_{i_0 i_1} \cdot a_{i_1 i_2} \cdot \dots \cdot a_{i_{k-1} i_k} \neq 0$$

Perron-Frobenius: A regular matrix $A$ has a unique eigenvector whose entries are all positive. The corresponding eigenvalue is simple and is the largest eigenvalue of $A$.

Lemma (Bochner method). Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$, and let $x \in \mathbb{R}^n_{>0}$. If

(1). The Hessian of $f$ at $x$ is regular, and for all $i$,

(2). $\partial_i f(x) > 0$, and

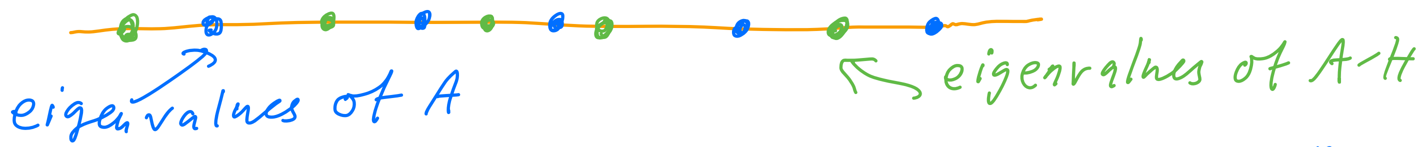
(3). The Hessian of $\partial_i f$ at $x$ has exactly one positive eigenvalue,

then the Hessian of $f$ at $x$ has exactly one positive eigenvalue.

# Proof of "Bochner method lemma":

A version of Cauchy interlacing (rank one perturbations) is

<u>Lemma C</u>: If $A$ is a symmetric matrix and $H$ is a rank one PSD matrix, then the eigenvalues of $A-H$ interlaces left of the eigenvalues of $A$:




Notation: $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_n f)$, $\nabla^2 f = (\partial_i \partial_j f)_{i,j=1}^n$

<u>Lemma D</u>: Suppose $g \in \mathbb{R}[x_1, \ldots, x_n]$ is homogeneous of degree $d$, and $g(x) > 0$. If $\nabla^2 g(x)$ has exactly one positive eigenvalue, then

$$d \cdot g \cdot \nabla^2 g - (d-1) \nabla g (\nabla g)^T \quad \text{(evaluated at } x\text{)}$$

is negative semidefinite.

<u>Proof</u>: Recall Euler's formula for homogeneous functions: $d \cdot g = \sum_{i=1}^n x_i \partial_i g$. This gives

(a) $$x^T \nabla^2 g \, x = \sum_{i,j} x_i x_j \partial_i \partial_j g = \sum_i x_i \sum_j x_j \partial_j (\partial_i g) \overset{E.F}{=}$$

$$= \sum_i x_i (d-1) \partial_i g = d(d-1) g$$

(b) $$x^T \nabla g (\nabla g)^T x = \sum_{i,j} x_i x_j \partial_i g \, \partial_j g = d^2 g^2$$

If $\nabla^2 g(x)$ has exactly one positive eigenvalue and $g(x) > 0$, then the matrix

$$B_\mu = d \cdot g \cdot \nabla^2 g - \mu \nabla g (\nabla g)^T \quad , \mu > 0$$

has at most one pos. eigenvalue by Lemma C (and exactly one pos. eigenvalue for $\mu > 0$ sufficiently small).

By (a) and (b), $x^T B_\mu x = d^2(d-1) g^2 - \mu d^2 g^2$.

Hence when $\mu \le d-1$, $B_\mu$ is NSD. $\square$

- Recall the <u>Löwner order</u> on symmetric matrices:

$$A \preceq B \quad \text{if} \quad B - A \text{ is PSD.}$$

<u>Proof (of Bochner method)</u>: For ease of notation let $f_i = \partial_i f$. (All polynomials below are evaluated at $x$).

By Lemma D:

$$(d-1) f_i \nabla^2 f_i \preceq (d-2) \nabla f_i (\nabla f_i)^T \qquad (\star)$$

Then by Euler's formula,

$$(d-2) \nabla^2 f = \left( (d-2) \partial_k \partial_l f \right)_{k,l} = \left( \sum_i x_i \partial_i \partial_k \partial_l f \right)_{k,l}$$

$$= \sum_i x_i \nabla^2 f_i \overset{(\star)}{\leq} \sum_i \frac{x_i}{f_i} \frac{(d-2)}{(d-1)} \nabla f_i (\nabla f_i)^T$$

We may rewrite this inequality as

$$(d-1)\nabla^2 f \leq (\nabla^2 f) D (\nabla^2 f) \qquad (*)$$

where $D$ is the diagonal matrix
$D = \mathrm{diag}(x_1/f_1, x_2/f_2, \ldots, x_n/f_n)$.
If $A = D^{1/2}(\nabla^2 f) D^{1/2}$, then $(*)$ translates as

$$A^2 - (d-1)A \geq 0.$$

Hence if $\lambda$ is an eigenvalue of $A$, then

$$0 \leq \lambda^2 - (d-1)\lambda = \lambda(\lambda - (d-1)),$$

i.e., no eigenvalue of $A$ lies in the open interval $(0, d-1)$.

Consider the vector $v = D^{-\frac{1}{2}} x = \left( x_i^{\frac{1}{2}} f_i^{\frac{1}{2}} \right)_i$.

Then

$$\begin{aligned} Av &= D^{\frac{1}{2}} (\nabla^2 f) D^{\frac{1}{2}} D^{-\frac{1}{2}} x = D^{\frac{1}{2}} (\nabla^2 f) x \\ &= D^{\frac{1}{2}} \Big( \sum_j x_j \partial_j \partial_i f \Big)_i = (d-1) D^{\frac{1}{2}} (f_i)_i \\ &= (d-1) \left( \frac{x_i^{\frac{1}{2}}}{f_i^{1/2}} \cdot f_i \right)_i = (d-1) v \end{aligned}$$

Since $\nabla^2 f$ is regular, so is $A$.

Hence by Perron-Frobenius, $v$ is the unique positive eigenvector corresponding to the maximum eigenvalue (which is $d-1$) of $A$. Since $A$ has no eigenvalue in $(0, d-1)$, we conclude that the matrix

$$A = D^{\frac{1}{2}}(\nabla^2 f)D^{\frac{1}{2}}$$

has exactly one positive eigenvalue. By Sylvester, so does $\nabla^2 f$. $\square$

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**Theorem** (B., Huh). Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be homog. of degree $d$. Then $f$ is Lorentzian iff

(M). $\mathrm{supp}(f)$ is M-convex, and

(L). $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalue.

Recall:

**Lemma** (Bochner method). Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$, and let $x \in \mathbb{R}^n_{>0}$. If

(1). The Hessian of $f$ at $x$ is regular, and for all $i$,

(2). $\partial_i f(x) > 0$, and

(3). The Hessian of $\partial_i f$ at $x$ has exactly one positive eigenvalue,

then the Hessian of $f$ at $x$ has exactly one positive eigenvalue.

Proof of thm:

$\Rightarrow$). Assume $f$ is Lorentzian. Then $D_{v_1} \cdots D_{v_{d-2}} f$ has Hessian with exactly one pos. eigenvalue. But then by continuity $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-2}} f$ is either $\equiv 0$, or has at most one positive eigenvalue. This gives (L).

(M). Induction over $d \geq 2$. If $d = 2$, then $f$ is stable, so $\mathrm{supp}(f)$ is M-convex. By definition Lorentzian polynom are closed under $D_v f$, $v \in \mathbb{R}^n_{>0}$,

But the Lorentzian property is closed

so $\partial_i f$ is either identically zero or Lorentzian if $f$ is Lorentzian.
Hence the supports of Lorentzian pol. are closed under $\partial_i M$ and $\tau M$.
(M) follows from Lemma R. (by induction)

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Other direction next time.