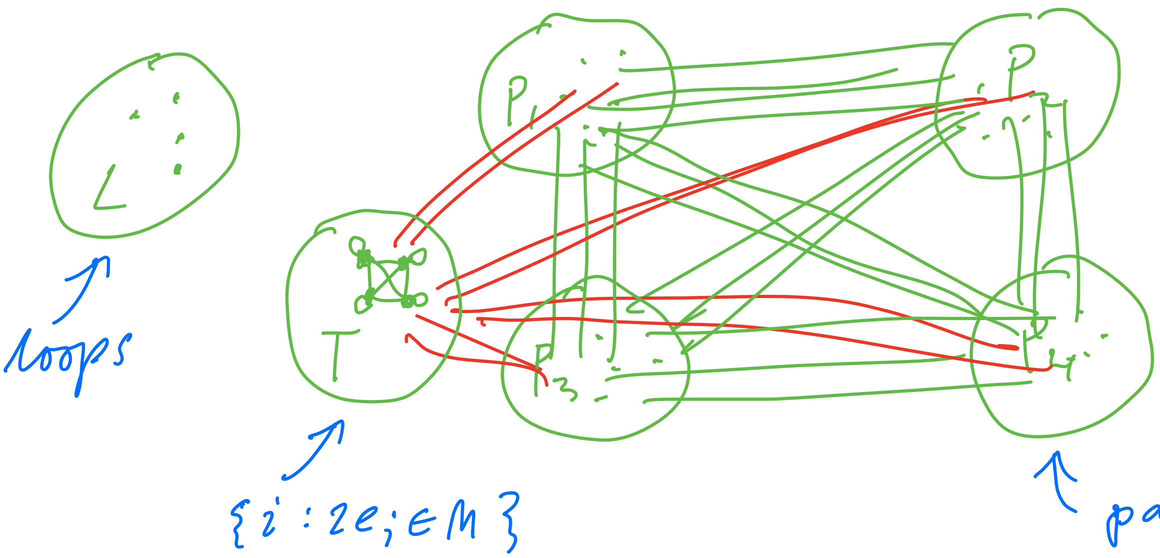
Recap

M-convex of rank 2 $\supseteq$ $\left\{\begin{array}{c}\text{supports of} \\ \text{quadratic Lorentzians} \\ \text{(stable)}\end{array}\right\}$



edge means $e_i + e_j \in M$

$\Rightarrow$ if f is a quadratic Lorentzian and $\partial_i f \not\equiv 0$ for all i, then the Hessian of f is regular.

Theorem M (B., Huh). Suppose $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is homogeneous of degree d. Then f is Lorentzian iff

(M) $\operatorname{supp}(f)$ is M-convex, and

(L) for all $i_1, i_2, \ldots, i_{d-2}$, the Hessian of $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalue

---

(L) is equivalent to:

(S) for all $i_1, \ldots, i_{d-2}$, $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ is stable.

<u>Def</u>: f is C-Lorentzian if for all $v_1, v_2, \dots, v_d \in C$,

(P) $D_{v_1} D_{v_2} \cdots D_{v_d} f > 0$

(AF) $(D_{v_1} \cdots D_{v_d} f)^2 \geq (D_{v_1} D_{v_1} D_{v_3} \cdots D_{v_d} f) \cdot (D_{v_2} D_{v_2} D_{v_3} \cdots D_{v_d} f)$.

We can replace (AF) with an equivalent condition:

(L) The Hessian of $D_{v_1} D_{v_2} \cdots D_{v_{d-2}} f$ has exactly one pos. eigenvalue.

- If $C = \mathbb{R}^n_{>0}$, then we say that f is Lorentzian.

Lemma (Bochner method). Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$, and let $x \in \mathbb{R}^n_{>0}$. If

(1). The Hessian of f at x is regular, and for all i,

(2). $\partial_i f(x) > 0$, and

(3). The Hessian of $\partial_i f$ at x has exactly one positive eigenvalue,

then the Hessian of f at x has exactly one positive eigenvalue.

Proof of $(\Leftarrow)$ in Theorem M:

- (M) and (L) implies Lorentzian:

$L' = \{$ polynomials satisfying (M) and (L) $\}$

$L = \{$ Lorentzian pol. $\}$

Prove $L' \subseteq L$ by induction over d = degree

$\underline{d=2}$ Trivial

Assume $d \geq 3$: We want to prove that $D_{v_1} D_{v_2} \cdots D_{v_{d-2}} f$ has exactly one pos. eigenvalue if $f \in L'$.

This Hessian is equal the Hessian of

$\sim \; g = D_{v_1} D_{v_2} \cdots D_{v_{d-3}} f$ at $x = v_{d-2}$

Apply Bochner method:

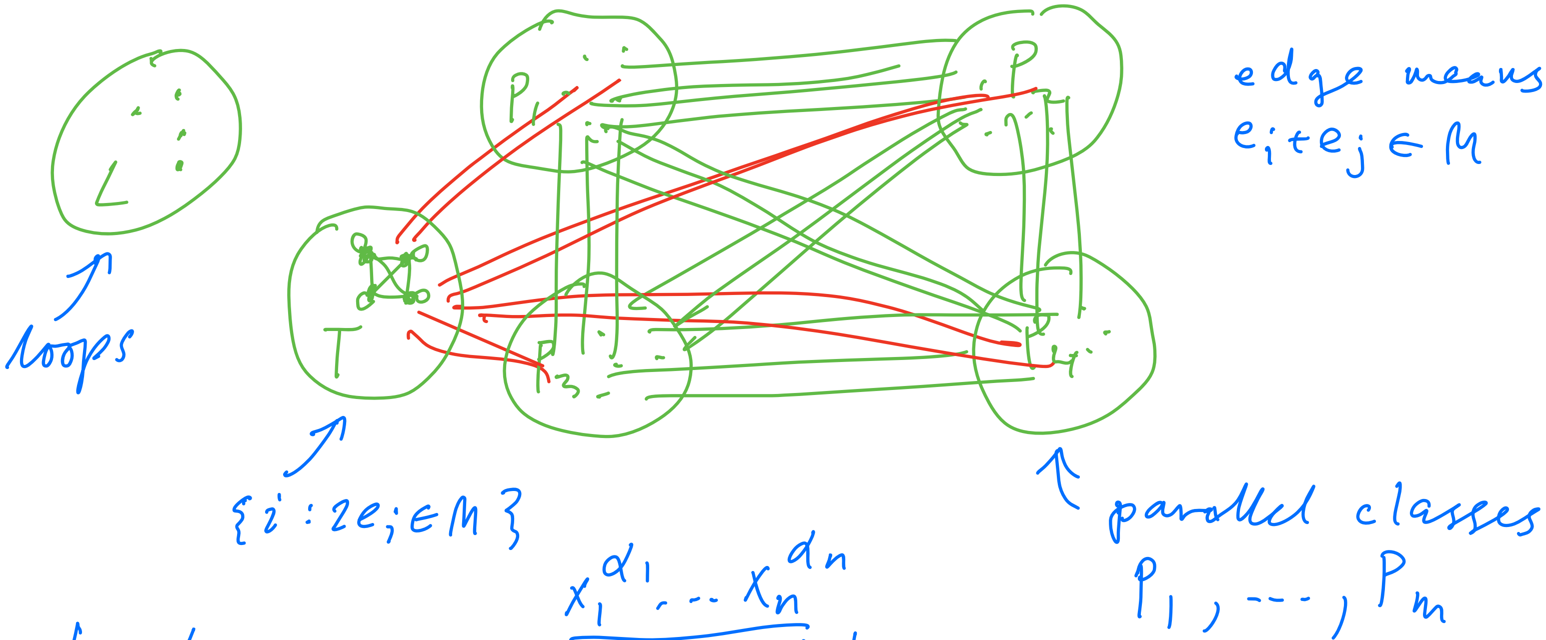
$$\partial_i g = \partial_i D_{v_1} \cdots D_{v_{d-3}} f =$$
$$= D_{v_1} \cdots D_{v_{d-3}} \partial_i f$$

By definition $\partial_i f \in L'$, so by induction $\partial_i f \in L$. So $\partial_i g = D_{v_1} \cdots D_{v_{d-3}} \partial_i f$ is Lorentzian. The Hessian of $\partial_i g$ has exactly one pos. eigenvalue.

The Hessian of g at $x \in \mathbb{R}^n_{>0}$ is regular, since support of the Hessian is equal to the support of $\tau^{d-2} \operatorname{supp}(f)$

$\leftarrow$ M-convex

So Bochner implies that The Hessian of g at x has exactly one pos. eigenvalue. $\square$



Implies: $\frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}$

$$p_M = \sum_{\alpha \in M} \frac{x^\alpha}{\alpha!} = \sum_{i \in T} \frac{x_i^2}{2} + \sum_{\substack{e_i + e_j \in M \\ i \neq j}} x_i x_j =$$

$$= \frac{1}{2}\left(\left(\sum_{e \in E \setminus L} x_e \right)^2 - \left(\sum_{e \in P_1} x_e \right)^2 - \dots - \left(\sum_{e \in P_m} x_e \right)^2 \right)$$

Thus, the Hessian of p_M is NSD on the hyperplane $\sum_{e \in E \setminus L} x_e = 0$.

Cauchy $\Rightarrow$ Hessian has at most one pos. eigenvalue $\Rightarrow$ Hessian has exactly one positive eigenvalue, since $p_M(1, \dots, 1) > 0$.

$\therefore$ p_M is Lorentzian, when $r = 2$.

Theorem (B., Huh) If M is M-convex, then p_M is Lorentzian

Proof. Notice that $\partial_i p_M = p_{\partial_i M}$

By Theorem M, we need to check (M) and (L).

(M) is for free.

(L) $\partial_{i_1} \cdots \partial_{i_{d-2}} p_M = p_{\partial_{i_1} \cdots \partial_{i_{d-2}} M}$ has at most one pos. eigenvalue since $\partial_{i_1} \cdots \partial_{i_{d-2}} M$ is of rank 2.

$\square$

- This characterises M-convex sets and matroids.

Mason's conjecture

- Let M be a matroid on E, $|E| = n$.
- $I \subseteq E$ is independent if I is a subset of some basis.
- Let f_k = # independent sets of size k

Mason's strongest conjecture:

$$\frac{f_k^2}{\binom{n}{k}^2} \geq \frac{f_{k-1}}{\binom{n}{k-1}} \cdot \frac{f_{k+1}}{\binom{n}{k+1}}, \quad 0 < k < n$$

Define $\quad I_M(x) = \sum_{I \text{ ind.}} x_0^{n-|I|} \prod_{e \in I} x_e$

We will prove that $I_M(x)$ is Lorentzian.

Given that, let

$u = (0, 1, 1, \ldots, 1)$, $v = (1, 0, \ldots, 0)$. Then

$$I_M(su + tv) = \sum_{k=0}^{n} f_k t^{n-k} s^k.$$

(AF) now implies Mason's conjecture.

Theorem (B., Huh) (Anari et al). $I_M(x)$ is Lorentzian.

Proof. By Theorem M, we need to prove (M) and (LI).

For free

By induction over rank.

$e \in E$: $\partial_{x_e} I_M(x) = I_{M/e}(x)$

contraction.

In checking that $\partial_{i_1} \cdots \partial_{i_{d-2}} I_M$ has exactly one pos. eigenvalue, this reduces it by induction to just consider $\partial_0^{n-2} I_M$. (Assume no loops)

$$\partial_0^{n-2} I_M = \frac{n!}{2} x_0^2 + (n-1)! \left(\sum_{e \in E} x_e\right) x_0$$

$$+ (n-2)! \sum_{\{i,j\} \in I} x_i x_j$$

Divide by $(n-2)!$

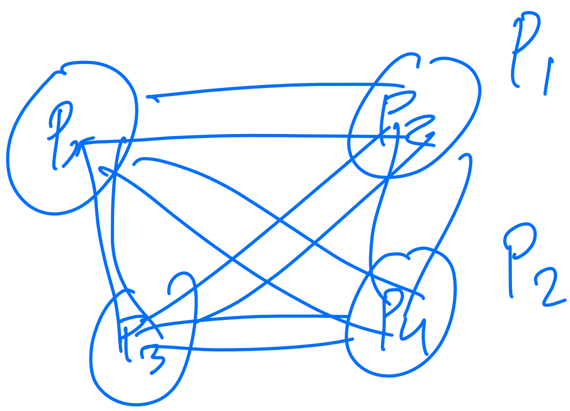
We want to prove that

$$\frac{n(n-1)}{2} x_0^2 + (n-1)\Big(\sum_{e\in E} x_e\Big) x_0 + \sum_{\{i,j\}\in I} x_i x_j$$

is stable. By characterization of stable quadratics this amounts to proving:

$$(\ast) \quad (n-1)^2 \Big(\sum_{e\in E} x_e\Big)^2 \geq 4\,\frac{n(n-1)}{2} \sum_{\{i,j\}\in I} x_i x_j$$

$\forall\, x \in \mathbb{R}^n$.



$$y_i = \sum_{e\in P_i} x_e \qquad i = 1, \dots, m, \quad m \leq n$$

$(\ast)$ translates as:

$$e_1(y)^2 \geq 2\,\frac{n}{n-1}\, e_2(y)$$

Newton's inequalities:

$$e_1(y)^2 \geq 2\frac{m}{m-1} e_2(y)$$

We can assume $e_2(y) > 0$.

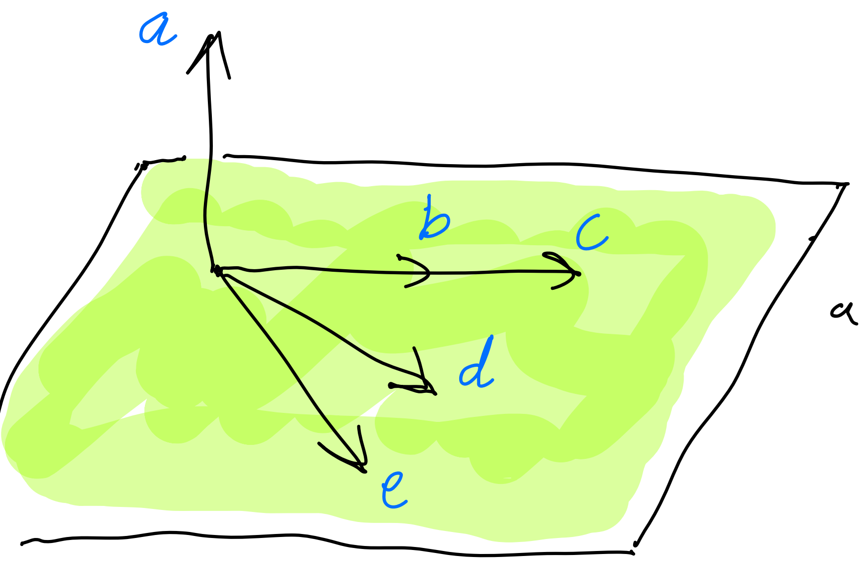
Since $m \leq n$,

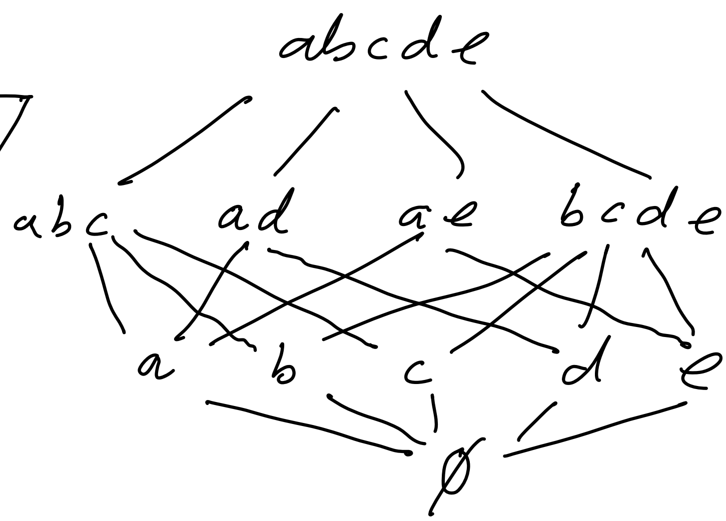
$$e_1(y)^2 \geq 2\frac{m}{m-1} e_2(y)$$
$$\geq 2\frac{n}{n-1} e_2(y)$$

$\square$

Heron-Rota-Welsh conjecture

- Let M be a matroid on E.
- $F \subseteq E$ is a flat if $r(F \cup e) > r(F)$, $\forall e \in E \setminus F$
- $\mathcal{L}(M) = \{\text{flats}\}$ is a geometric lattice, under set inclusion





- Recall the characteristic polynomial

$$\chi_M(t) = \sum_{F \in \mathcal{L}(M)} \mu(\emptyset, F)\, t^{r(M) - r(F)}$$

$$= \sum_{k=0}^{r(M)} (-1)^k w_k t^{r(M)-k}, \text{ where } w_k \geq 0$$

are the Whitney numbers of the 2^{nd} kind.

Theorem (Adiprasito–Huh–Katz).

$w_k^2 \geq w_{k-1} w_{k+1}$, $0 < k < r(M)-1$

We will give a Lorentzian proof (B., Leake)

- For each pair $F < K \in \mathcal{L}(M)$, we will define a polynomial

$$\mathrm{pol}_K^L(t) \in \mathbb{R}[t_F : K < F < L]$$

which we will prove is C-Lorentzian.

- Let $\mathcal{E}_K^L = \{(y_S)_{K \subset S \subset L} : y_S \in \mathbb{R}\}$, a Euclidean space of $\dim = 2^{|L \setminus K|} - 2$

- $\mathcal{M}_K^L$ is the subspace of $\mathcal{E}_K^L$ consisting of modular elements

$$y_S + y_T = y_{S \cup T} + y_{S \cap T} \ , \quad y_K = y_L = 0$$

Fact: $y \in \mathcal{M}_K^L \iff$ there are real numbers x_i, $i \in L \setminus K$ s.t.

$$y_S = \sum_{i \in S \setminus K} x_i \quad \text{and} \quad \sum_{i \in L \setminus K} x_i = 0$$

- C_L^K is the open convex cone in $\mathcal{E}_L^K$ consisting of all strictly submodular elements:

$$y_S + y_T > y_{S\cup T} + y_{S\cap T}, \quad y_K = y_L = 0,$$

for all incomparable $S, T \in (K, L)$