Lorentzian polynomials on the cone of strictly submodular functions, and the Heron-Rota-Welsh conjecture.

(joint with Jonathan Leake).

lattice of flats in M.

- For each pair $K < L \in \mathcal{L}(M)$, we will define a polynomial
$$\mathrm{pol}_K^L(t) \in \mathbb{R}[t_F : K < F < L]$$
which we will prove is C-Lorentzian.

- Let $\mathcal{E}_K^L = \{ (y_S)_{K \subset S \subset L} : y_S \in \mathbb{R} \}$, a Euclidean space of $\dim = 2^{|L \setminus K|} - 2$

- $\mathcal{M}_K^L$ is the subspace of $\mathcal{E}_K^L$ consisting of modular elements
$$y_S + y_T = y_{S \cup T} + y_{S \cap T}, \quad y_K = y_L = 0$$

<u>Fact</u>: $y \in \mathcal{M}_K^L \iff$

there are real numbers x_i, $i \in L \setminus K$ s.t.
$$y_S = \sum_{i \in S \setminus K} x_i \quad \text{and} \quad \sum_{i \in L \setminus K} x_i = 0$$

- C_L^K is the open convex cone in $\mathcal{E}_L^K$ consisting of all strictly submodular elements:

$$y_S + y_T > y_{S\cup T} + y_{S\cap T}, \quad y_K = y_L = 0,$$

for all incomparable $S, T \in (K, L)$

- The lineality space of an open convex cone C in $\mathbb{R}^n$ is

$$L(C) = \overline{C} \cap (-\overline{C})$$

(the largest linear space in $\overline{C}$)

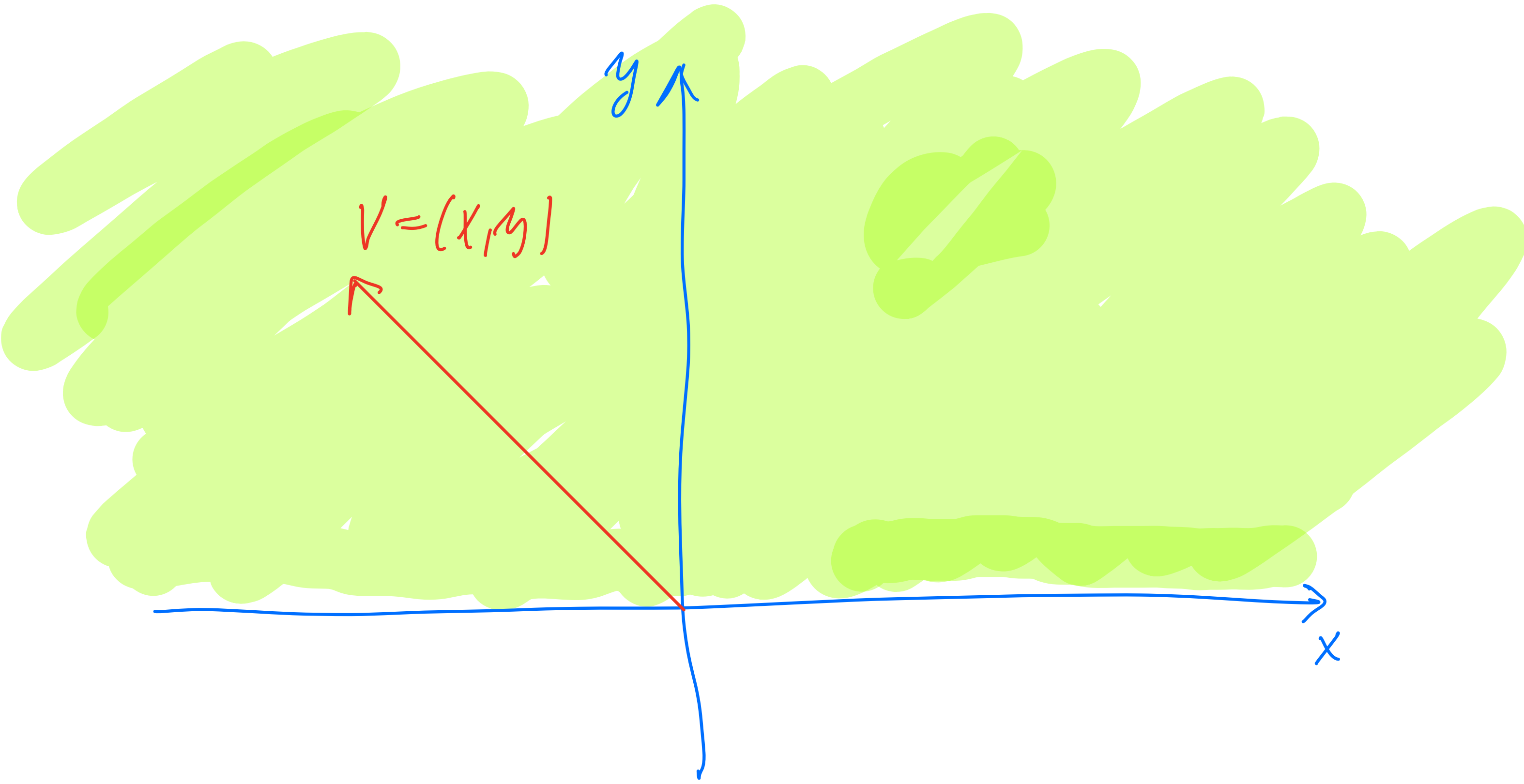
<u>EX</u>. The lineality space of C_K^L is $\mathcal{M}_K^L$.

- C is effective if

$$C = C \cap \mathbb{R}^n_{>0} + L(C)$$

Ex. $C = \{(x,y)\in\mathbb{R}^2 : y>0\}$

$L(C) = \{y=0\}$



$$v = (x,y) = \underset{C\cap\mathbb{R}^2_{\geq 0}}{\underset{\cap}{(1,y)}} + \underset{L(C)}{\underset{\cap}{(x-1,0)}}$$

Prop. C_K^L is effective.

Projections: If $K \subseteq F \subset G \subseteq L$, define a projection $\pi_F^G : \mathcal{E}_K^L \longrightarrow \mathcal{E}_F^G$ by

$$\pi_F^G(t) = \left(t_S - t_F \frac{|G \setminus S|}{|G \setminus F|} - t_G \frac{|S \setminus F|}{|G \setminus F|} \right)_{F \subset S \subset G}$$

write $\pi_G^F(t) = (y_S)_S$

$$y_F = t_F - t_F - t_G \cdot 0 = 0$$

$$y_G = t_G - 0 - t_G = 0$$

Lemma: $\pi_F^G(C_K^L) \subseteq C_F^G$

$\pi_F^G(M_K^L) \subseteq M_F^G$

- If $K \subseteq A \subseteq B \subseteq C \subseteq D \subseteq L$

$$\begin{tikzcd}
\mathcal{E}_K^L \arrow[r, "\pi_B^C"] \arrow[d, "\pi_A^D"'] & \mathcal{E}_B^C \\
\mathcal{E}_A^D \arrow[ur, "\pi_B^C"'] &
\end{tikzcd}$$

commutes

<u>Definition</u>: Let $d(K,L) = r(L) - r(K) - 1$.

Define $pol_K^L(t) \in \mathbb{R}[F : K < F < L]$ recursively by

- $pol_K^L(t) \equiv 1$ if $d(K,L) = 0$
  (L covers K in $\mathcal{L}(M)$)

- If $d(K,L) \geq 1$, then

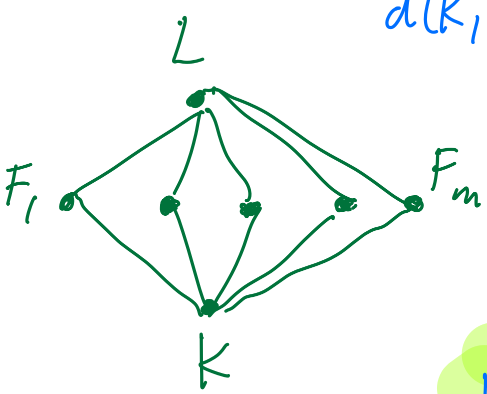
$$d(K,L) \cdot pol_K^L(t) = \sum_{K<F<L} t_F \cdot pol_K^F(\pi_K^F(t)) \cdot pol_F^L(\pi_F^L(t))$$

- By induction it follows that the monomials in the support are of the form
  $t_{F_1} t_{F_2} \cdots t_{F_d}$ where $F_1 \leq F_2 \leq \cdots \leq F_d$

- $\deg(pol_K^L(t)) = d(K,L)$

- <u>$d(K,L) = 1$</u>:

$$\overset{=1}{d(K,L)} \cdot pol_K^L(t) = \sum_{K<F<L} t_F \cdot \overset{=1}{pol_K^F(\pi_K^F(t))} \cdot \overset{=1}{pol_F^L(\pi_F^L(t))}$$



$$pol_K^L(t) = \sum_{K<F<L} t_F$$

- Recall that $F_1 \setminus K, \dots, F_m \setminus K$ partitions $L \setminus K$.

Also, if $w \in \mathcal{M}_K^L$, then $w_S = \sum_{i \in S \setminus K} x_i$, $\sum_{i \in L \setminus K} x_i = 0$.

Thus

$$\mathrm{pol}_K^L(w) = \sum_{j=1}^{m} \Big(\sum_{i \in F_j \setminus K} x_i \Big) = \sum_{i \in L \setminus K} x_i = 0$$

- Hence $\mathrm{pol}_K^L(w+t) = \mathrm{pol}_K^L(t)$, $\forall\, t \in \mathcal{E}_K^L$, $w \in \mathcal{M}_K^L$

- Since C_K^L is effective

$$\mathrm{pol}_K^L(v) > 0 \quad \forall\ v \in C_K^L$$

Recall that if $f \in \mathbb{R}[x_1, \dots, x_n]$ is homogeneous of degree d, then

$$d \cdot f = \sum_{i=1}^{n} x_i \partial_i f \qquad \text{(Euler's formula)}$$

The definition

$$d(K,L) \cdot \mathrm{pol}_K^L(t) = \sum_{K < F < L} t_F \cdot \underbrace{\mathrm{pol}_K^F(\pi_K^F(t)) \cdot \mathrm{pol}_F^L(\pi_F^L(t))}_{= \frac{\partial}{\partial t_F} \mathrm{pol}_K^L(t)}$$

suggests:

Lemma D: If $K < L \in \mathcal{L}(M)$, then

$$\frac{\partial}{\partial t_F} \mathrm{pol}_K^L(t) = \mathrm{pol}_K^F(\pi_K^F(t)) \cdot \mathrm{pol}_F^L(\pi_F^L(t))$$

Lemma LS: $\mathrm{pol}_K^L(t+w) = \mathrm{pol}_K^L(t)$

$\forall\ t \in \mathcal{E}_K^L$ and $w \in \mathcal{M}_K^L$

(This means that we can always assume that $v_s > 0$ for all $v = (v_s)_{K<s<L} \in C_K^L$)

Lemma P: If $v_1, \dots, v_d \in C_K^L$, $(d = d(K,L))$

then

(1). $D_{v_1} D_{v_2} \cdots D_{v_d} \mathrm{pol}_K^L(t) > 0$

(2). The Hessian of $D_{v_1} \cdots D_{v_{d-2}} \mathrm{pol}_K^L(t)$ is regular.

regularity follows semimodularity of $\mathcal{L}(M)$.

Theorem E (B., Leake). Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be homogeneous of degree $d \geq 3$, and C an effective cone.

If for all $v_1, \dots, v_d \in C$,

(1). $D_{v_1} D_{v_2} \cdots D_{v_d} f > 0$, and

(2). The Hessian of $D_{v_1} D_{v_2} \cdots D_{v_{d-2}} f$ is regular, and

(3). $\partial_i f$ is C-Lorentzian for all i,

then f is C-Lorentzian.

Proof: Apply Bochner method lemma.

Theorem (B., Leake). $\mathrm{pol}_K^L(t)$ is C_K^L-Lorentzian.

Proof: Properties (1) and (2) of Theorem E are satisfied (by Lemma P).

The proof is now by induction on $d(K,L) \geq 2$.

Recall

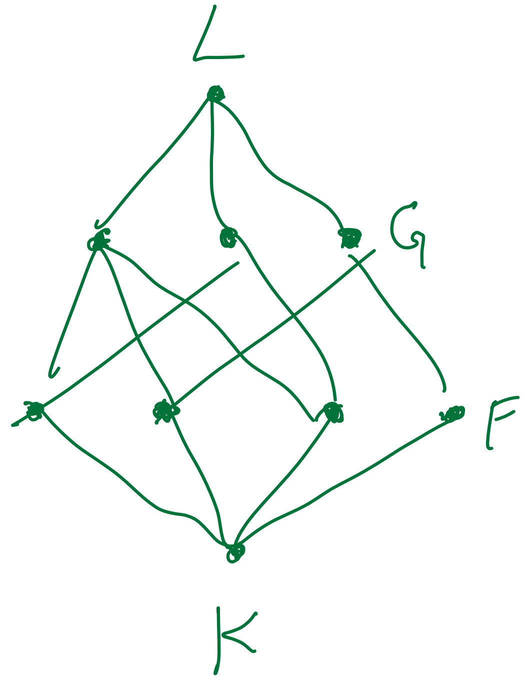
$$\frac{\partial}{\partial t_F} \mathrm{pol}_K^L(t) = \mathrm{pol}_K^F(\pi_K^F(t)) \cdot \mathrm{pol}_F^L(\pi_F^L(t))$$

- $\mathrm{pol}_K^F(t)$ and $\mathrm{pol}_F^L(t)$ are C_K^F-Lorentzian and C_F^L-Lorentzian by induction.
- Then $\mathrm{pol}_K^F(\pi_K^F(t))$ and $\mathrm{pol}_F^L(\pi_F^L(t))$ are C_K^L-Lorentzian (since $\pi_A^B(C_K^L) \subseteq C_A^B$)
- C-Lorentzian polynomials are closed under products.

$\therefore$ $\frac{\partial}{\partial t_F} \mathrm{pol}_K^L(t)$ is C_K^L-Lorentzian.

(3) follows.

It remains to consider $d(K,L) = 2$.



Denote by F, the flats of rank 1

—||— G, —||— 2

A computation gives:

$$2 \cdot pol_K^L(t) = \Big(\sum_F t_F\Big)^2 - \sum_G \Big(t_G - \sum_{F<G} t_F\Big)^2$$

$pol_K^L(t)$ is NSD on the hyperplane $\sum_F t_F = 0$

- Hence by Cauchy interlaing, the Hessian has at most one positive eigenvalue.
- Since $pol_K^L(v) > 0$ for $v \in C_K^L$, the Hessian has exactly one positive eigenvalue.

$$= \frac{1}{d!} \underbrace{D_v D_v \ldots D_v}_{d \text{ time}} f > 0$$

$\square$

Heron-Rota-Welsh conjecture

- Let $K = \emptyset$, $L = E$ (no loops)
- $\alpha, \beta \in \overline{C_\emptyset^E}$ are defined by

$$\alpha = \left(\frac{|S|}{|E|}\right)_{\emptyset \subset S \subset E} \qquad \beta = \left(\frac{|E \setminus S|}{|E|}\right)_{\emptyset \subset S \subset E}$$

<u>Theorem</u>: If we write

$$\mathrm{pol}_\emptyset^E(s\alpha + t\beta) = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} \binom{r-1}{k} a_k s^{r-1-k} t^k,$$

then a_k is the k^{th} coefficient of the polynomial

$$f(t) = \frac{w_0 + w_1 t + \dots + w_r t^r}{1+t}$$

$$\boxed{\chi_M(t) = \sum_{i=0}^{r} w_i (-1)^{r-i} t^{r-i}}$$

<u>Theorem</u>: $\{w_k\}_{k=0}^r$ is log-concave.

<u>Proof</u>: $f(t)$ log-concave and $g(t)$ log-concave, then $f(t)g(t)$ log-concave.

(AF) $\Rightarrow$ (a_k) is log-concave

$\Rightarrow$ (w_k) — // —

$\square$