Notes on a combinatorial formula for the nabla operator

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Let $\mathbf{x} = \{\mathbf{x_1}, ..., \mathbf{x_n}\}$ be n letters in the variable x, and similarly for other bold faced fonts. In the talk I considered the image M of the map

$$\pi : \mathbb{C}[\mathbf{x}, \mathbf{z}] \to \bigoplus_{\sigma \in S_n} \mathbb{C}[\mathbf{x}] \cdot e_\sigma$$

where

$$\pi(f(\mathbf{x}, \mathbf{z})) = \sum_{\sigma \in S_n} f(\mathbf{x}, \mathbf{x}_\sigma) e_\sigma.$$

For instance, when $x_n = 3$, M has a free basis given by $\mathcal{B} = \{f_\sigma : \sigma \in S_3\}$, where

$$f_{1,2,3} = e_{1,2,3} + e_{1,3,2} + e_{2,1,3} + e_{2,3,1} + e_{3,1,2} + e_{3,2,1}$$
$$f_{1,3,2} = (x_3 - x_2)\, e_{1,3,2} + (x_3 - x_1)\, e_{2,3,1} + (x_3 - x_2)\, e_{3,1,2} + (x_3 - x_1)\, e_{3,2,1}$$
$$f_{2,1,3} = (x_2 - x_1)\, e_{2,1,3} + (x_2 - x_1)\, e_{2,3,1} + (x_3 - x_1)\, e_{3,1,2} + (x_3 - x_1)\, e_{3,2,1}$$
$$f_{2,3,1} = (-x_2 + x_1)(-x_3 + x_1)\, e_{2,3,1} + (-x_2 + x_1)(-x_3 + x_1)\, e_{3,2,1}$$
$$f_{3,1,2} = (-x_3 + x_1)(-x_3 + x_2)\, e_{3,1,2} + (-x_3 + x_1)(-x_3 + x_2)\, e_{3,2,1}$$
$$f_{3,2,1} = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1)\, e_{3,2,1}$$

- The module M is an algebraic presentation for the equivariant cohomology $M = H_T^*(Fl_n)$ of the complex flag variety $Fl_n = GL_n/B$, and the torus $T = (\mathbb{C}^*)^n \subset GL_n$ is the maximal torus. The inclusion map

$$M \hookrightarrow \bigoplus_\sigma \mathbb{C}[\mathbf{x}] e_\sigma$$

is identified with the pullback of the inclusion map of the fixed point set, which is S_n:

$$i^* : H_T^*(Fl_n) \hookrightarrow H_T^*(Fl_n^T) = \bigoplus_{p \in Fl_n^T} H_T^*(p) = \bigoplus_\sigma \mathbb{C}[\mathbf{x}] e_\sigma$$

- Since Fl_n is a GKM space, this will imply that M is free over $\mathbb{C}[\mathbf{x}]$. Since Fl_n has a T-equivariant CW decomposition into affine space which are the Schubert cells, it has a corresponding basis which are upper-triangular with respect to e_σ in the Bruhat order. The basis f_σ is that basis.

- The ring structure is given by multiplication pointwise:

$$\left(\sum_\sigma a_\sigma(\mathbf{x})e_\sigma\right)\cdot\left(\sum_\sigma b_\sigma(\mathbf{x})e_\sigma\right)=\left(\sum_\sigma a_\sigma(\mathbf{x})b_\sigma(\mathbf{x})e_\sigma\right).$$

  The identity element is the sum of all the basis vectors, which is $1 = f_{1,2,3}$ above in the case of $n = 3$.

- There is an action of multiplication by $\mathbb{C}[\mathbf{z}]$ on M that is compatible with π, given by

$$z_i \cdot e_\sigma = x_{\sigma_i} e_\sigma.$$

  It can be interpreted as multiplication by the Chern classes of the tautlogical bundles.

- There are two commuting actions of the symmetric group by

$$\sigma \cdot a(\mathbf{x})e_\tau = a(\mathbf{x}_\sigma)e_{\sigma\cdot\tau}, \quad \sigma * a(\mathbf{x})e_\tau = e_{\tau\cdot\sigma^{-1}}$$

  They are called "dot" and "star" respectively, following Tymoczko and Knutson's terminology.

- There is a characterization of M given by the GKM relations:

$$H_T^*(Fl_n) = \left\{\sum_\sigma a_\sigma(\mathbf{x})e_\sigma : a_\sigma(\mathbf{x}) - a_{s_{i,j}\sigma}(\mathbf{x}) \in (x_i - x_j)\right\}$$

  for any transposition $s_{i,j}$. During the lecture I attempted to verify that $f_{1,3,2}$ satisfies the GKM relations on the fly in my MAPLE code. Here is what the test should have looked like corresponding to the transposition $s_{1,3}$:

```
> f123;
(x3 - x2) e[1, 3, 2] + (x3 - x1) e[2, 3, 1] + (x3 - x2) e[3, 1, 2]

     + (x3 - x1) e[3, 2, 1]

> s13 := [3,2,1];
                                 s13 := [3, 2, 1]

> eval(f123,[seq(e[op(sig)]=e[op(s13[sig])],sig=getsn(3))]);
(x3 - x2) e[3, 1, 2] + (x3 - x1) e[2, 1, 3] + (x3 - x2) e[1, 3, 2]

     + (x3 - x1) e[1, 2, 3]

> factor(f123-%); eval(%,x1=x3);
        (-x3 + x1) (-e[2, 3, 1] - e[3, 2, 1] + e[2, 1, 3] + e[1, 2, 3])

                                        0

> 
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- The equivariant Borel-Moore homology $H_*^T(Fl_n)$ may be described as follows: let $\mathcal{B}' = \{f'_\sigma\}$ be the dual basis to f_σ, taken as a basis of the localization $\bigoplus_\sigma \mathbb{C}(\mathbf{x})$, which is a vector space over $\mathbb{C}(\mathbf{x})$. It is given by:

$$f'_{1,2,3} = e_{1,2,3},$$

$$f'_{1,3,2} = \frac{e_{1,2,3}}{-x_3+x_2} - \frac{e_{1,3,2}}{-x_3+x_2}$$

$$f'_{2,1,3} = \frac{e_{1,2,3}}{-x_2+x_1} - \frac{e_{2,1,3}}{-x_2+x_1}$$

$$f'_{2,3,1} = \frac{e_{1,2,3}}{(-x_2+x_1)(-x3+x_2)} - \frac{e_{1,3,2}}{(-x_2+x_1)(-x_3+x_2)} - \frac{e_{2,1,3}}{(-x_2+x_1)(-x_3+x_1)} + \frac{e_{2,3,1}}{(-x_2+x_1)(-x_3+x_1)}$$

$$f'_{3,1,2} = \frac{e_{1,2,3}}{(-x_2+x_1)(-x3+x_2)} - \frac{e_{1,3,2}}{(-x_3+x_1)(-x_3+x_2)} - \frac{e_{2,1,3}}{(-x_2+x_1)(-x_3+x_2)} + \frac{e_{3,1,2}}{(-x_3+x_1)(-x_3+x_2)}$$

$$f_{3,2,1} = \frac{e_{1,2,3}}{(-x_3+x_1)(-x3+x_2)(-x_2+x_1)} - \frac{e_{1,3,2}}{(-x_3+x_1)(-x_3+x_2)(-x_2+x_1)} - \frac{e_{2,1,3}}{(-x_3+x_1)(-x_3+x_2)(-x_2+x_1)} + \frac{e_{2,3,1}}{(-x_3+x_1)(-x_3+x_2)(-x_2+x_1)} + \frac{e_{3,1,2}}{(-x_3+x_1)(-x_3+x_2)(-x_2+x_1)} - \frac{e_{3,2,1}}{(-x_3+x_1)(-x_3+x_2)(-x_2+x_1)}.$$

  The fact that M is also isomorphic to $H_*^T(Fl_n)$, by multiplication by $\Delta = \prod_{i<j}(x_i - x_j)$ is related to Poincare duality, and would not apply for varieties which are not smooth or projective.

The purpose of describing this is that it is a simpler model for what happens with the module from the talk, which is

$$\pi : \mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}] \to \bigoplus_\sigma \mathbb{C}[\mathbf{x},\mathbf{y}] \cdot e_\sigma = \bigoplus_{\sigma,a} \mathbb{C}[\mathbf{x}]\mathbf{y}^a e_\sigma$$

On the one hand, it is a type of polygraph module in Haiman's machinery, but on the other it is the equivariant Borel-Moore homology of a certain space whose fixed points are given by $\mathbb{Z}^n \rtimes S_n$. In fact, we have

$$\nabla_X e_n\left[\frac{XY}{(1-q)(1-t)}\right] = \mathcal{F}_{X,Y} M = q^{-kn(n+1)/2} \omega_X \mathcal{F}_{X,Y} H_*^T(Y_{\gamma_1}^+)$$

where $Y_{\gamma_k}^+$ is a certain positive cone in the unramified affine Springer fiber, described in our follow up paper, "GKM spaces and the signed positivity of the nabla operator." The negative signs have to do with the fact that Borel-Moore homology is naturally graded in the negative direction, noting that the dual basis above have negative degree.