Stanley's Chromatic symmetric function (1995)

Def: Given a finite loopless simple graph $G = (V, E)$ and the set K_G of all proper coloring on V (i.e. $K_G = \{\kappa : V \to \mathbb{Z}^+ \mid \kappa(a) \neq \kappa(b) \ \forall \{a,b\} \in E\}$).

endpoints of every edge have distinct colors

Define $X_G(x_1, x_2, \ldots) := \sum_{\kappa \in K_G} x^{\kappa}$ where $x^{\kappa} = \prod_{v \in V} x_{\kappa(v)}$.

Fact: X_G is symmetric.

e.g. G :

κ : 5 7 5

$x^{\kappa} = x_5^2 x_7$

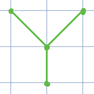
$X_G(x_1, x_2, \ldots) = 6m_{111} + m_{21}$

use 3 distinct colors ∴ 6 different ways to color 3 vertices with any chosen 3 colors

use only 2 colors, then the middle vertex must be of a different color, so the corresponding term must be $x_a^2 x_b$ or $x_a x_b^2$ with colors a and b.

In this case, $X_G(x_1, x_2, \ldots) = 4s_{111} + s_{21} = 3e_3 + e_{21}$ (e-positive and hence s-positive)

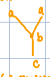
e.g. G :



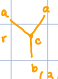
$X_G(x_1, x_2, \ldots) = 24m_{1111} + 6m_{211} + m_{31}$

4! ways to color with 4 colors

Given colors a, b, c, say a is used twice, then the "center" must be b or c and the other vertices must be of different color.



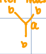
or



(3 choices for the position of c)

(3 choices for the position of b)

only 2 colors, say a, b, and the center must be colored differently.



$(x_a x_b^3)$

In this case, $X_G(x_1, x_2, \ldots) = s_{31} - s_{22} + 5s_{211} + 8s_{1111}$ (not s-positive, hence not e-positive either.)

Idea: Expand $X_G(x_1, x_2, \ldots)$ in terms of familiar bases for Λ. What does this have to do with G ?

Sample Theorem: (by Stanley)

Write $X_G(x_1, x_2, \ldots) = \sum_{\lambda \vdash |V|} \alpha_\lambda e_\lambda$. Then for any $m \in \mathbb{Z}^+$, $\sum_{\substack{\lambda \vdash |V| \\ \ell(\lambda) = m}} \alpha_\lambda =$ # acyclic orientations of G with exactly m sinks.

e.g. G :

$X_G(x_1, x_2, \ldots) = 3e_3 + e_{21}$

$m = 1$ $\sum_{\substack{\lambda \vdash 3 \\ \ell(\lambda) = 1}} \alpha_\lambda = \alpha_3 = 3$ →

$m = 2$ $\sum_{\substack{\lambda \vdash 3 \\ \ell(\lambda) = 2}} \alpha_\lambda = \alpha_{21} = 1$ →

$m = 3$ $\sum_{\substack{\lambda \vdash 3 \\ \ell(\lambda) = 3}} \alpha_\lambda = \alpha_{111} = 0$ (impossible to have 3 sinks in G).

* $X_G(x_1, x_2, \ldots)$ contains more information than the chromatic polynomial $X_G(t) =$ # proper colorings on G (V) with t colors.

Indeed, $X_G(n) = X_G(\underbrace{1, 1, 1, \ldots, 1}_{n}, 0, 0, \ldots\ldots)$

e.g.

$X_G(n) = n(n-1) \cdot (n-1)$

$X_G(1,1,1,1,0,0,\ldots,0) = 6m_{111}(1,1,1,1,0,0,\ldots) + m_{21}(1,1,1,1,0,0,\ldots)$

$x_1^2 x_2 x_3, x_1 x_2^2 x_3, x_1 x_2 x_3^2; x_1^2 x_2 x_4, x_1 x_2^2 x_4, x_1 x_2 x_4^2; x_1^2 x_3 x_4, x_1 x_3^2 x_4, x_1 x_3 x_4^2; x_2^2 x_3 x_4, x_2 x_3^2 x_4, x_2 x_3 x_4^2.$

$= 6 \cdot 4 + 12 = 36 = 4 \cdot 3 \cdot 3 = X_G(4).$

$x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4$

Open problem: It is known that any two trees with the same number of vertices share the same chromatic polynomial.

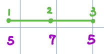
Are there any non-isomorphic trees share the same chromatic symmetric function ?

Given $G=(V,E)$ with $V=[n]=\{1,2,\dots,n\}$.

\# coloring preserving order of vertices on each edge

For $\kappa \in K_G$, define $\mathrm{asc}(\kappa) = |\{(i,j)\in E : i<j \text{ and } \kappa(i)<\kappa(j)\}|$

e.g. G :



κ :

$\mathrm{asc}(\kappa) = |\{(5,7)\}| = 1$

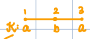
Define $X_G(x;t) := \sum_{\kappa\in K_G} t^{\mathrm{asc}(\kappa)} x^{\kappa}$

* $X_G(x;t)$ is a quasi-symmetric function. We call it a Chromatic quasi-symmetric function.

$X_G(x;1) = X_G(x_1,x_2,\dots)$ (Stanley's Chromatic symmetric function).

e.g. G :

κ : corresponds to $t^1 x^{\kappa} = t x_5^2 x_7$



κ : if $b>a$, then $\mathrm{asc}(\kappa)=|\{(1,2)\}|=1$

if $b<a$, then $\mathrm{asc}(\kappa)=|\{(2,3)\}|=1$

$X_G(x;t) = (\underbrace{1+4t+t^2}_{\text{Eulerian polynomial}})m_{111} + t\,m_{21}$ ($\therefore$ symmetric)

(coeff. of t^m = # $\sigma\in S_n$ with exactly m ascents)

coeff of t^0 = no ascent $\sigma\in S_3$ = 321

coeff of t = exactly 1 ascent = 132, 213, 231, 312

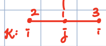
coeff of t^2 = exactly 2 ascents = 123

In this case, $X_G(x;t) = s_{111} + t(2s_{11}+s_{21}) + t^2 s_3 = e_3 + t(e_3+e_{21}) + t^2 e_3$ (e-positive and hence s-positive).

e.g. G :

$X_G(x;t) = (2+2t+2t^2)m_{111} + \sum_{i<j} x_i^2 x_j + t^2 \sum_{i<j} x_i x_j^2$ which is quasi-symmetric

1 3 2 / 2 3 1   1 2 3 / 3 2 1   2 1 3 / 3 1 2



$(j>i)$

$\therefore \mathrm{asc}(\kappa)=0$

$(j>i)$

$\therefore \mathrm{asc}(\kappa)=2$

Natural unit interval graph

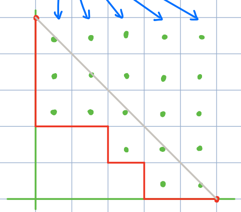
Def: $\vec{m}=(m_1,m_2,\dots,m_n)\in \mathbb{Z}^n_{>0}$ is a Hessenberg vector if

- $\vec{m}$ is weakly increasing
- $i\le m_i\le n$ $\forall i$ ($\therefore m_n = n$)

* {Hessenberg vector of length n} is in bijection with {Dyck paths with n horizontal steps}

i.e. stay below $x+y=n$

e.g. $\vec{m}=(3,3,4,5,5)$, $n=5$



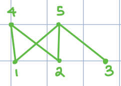
Define a poset $P_{\vec{m}}$ on $[n]$ where $i <_{P_{\vec{m}}} j$ iff $m_i < j$

e.g. $\vec{m}=(3,3,4,5,5)$

$\therefore 1 <_{P_{\vec{m}}} 4$, $1 <_{P_{\vec{m}}} 5$

$2 <_{P_{\vec{m}}} 4$, $2 <_{P_{\vec{m}}} 5$ $\Rightarrow P_{\vec{m}}$:

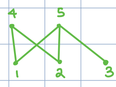
$3 <_{P_{\vec{m}}} 5$



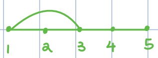
incomparable

Define $G_{\vec{m}} = \mathrm{Inc}(P_{\vec{m}})$

e.g. $\vec{m}=(3,3,4,5,5)$. $\Rightarrow P_{\vec{m}}$:



$\Rightarrow G_{\vec{m}} =$



* $P_{\vec{m}}$ is 3+1 free and 2+2 free. i.e. no subposet

or

Theorem: If $\vec{m}$ is a Hessenberg vector, then $X_G(x;t) \in \Lambda[t] = \Lambda_{\mathbb{C}[t]}$.

Idea of Proof: Show that $X_G(x;t)$ is invariant under each transposition $x_a \leftrightarrow x_{a+1}$.

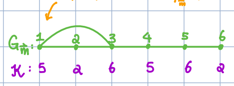
Lemma: Let $\vec{m}$ be a Hessenberg vector, $\mathcal{K} \in K_{G_{\vec{m}}}$, $a \in \mathbb{Z}_{>0}$. Define

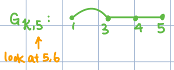
$G_{\mathcal{K},a} :=$ subgraph of $G_{\vec{m}}$ induced on $\mathcal{K}^{-1}(a) \cup \mathcal{K}^{-1}(a+1)$ (vertices colored a or $a+1$)

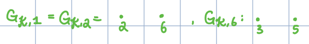
Then every connected component of $G_{\mathcal{K},a}$ is a path $i_1 - i_2 - \cdots - i_j$ with $i_1 < i_2 < \cdots < i_j$.

e.g. $\vec{m} = (3,3,4,5,6,6)$

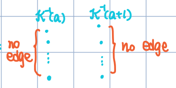
$m_1 < 4,5,6$ $\therefore$ $1 <_{P_{\vec{m}}} 4,5,6$ hence 1 is incomparable with 2, 3. Similarly, 2 is incomparable with 3.







Proof of Lemma: First note that $G_{\mathcal{K},a}$ is bipartite b/c $\mathcal{K}$ is a proper coloring:



Let $G_{\mathcal{K},a} = (V, E)$.

$\therefore$ $\{x,y\}, \{y,z\} \in E \Rightarrow \{x,z\} \notin E$.

Consider  . We want to show $x<y<z$ or $x>y>z$.

Case I) $x<y$:

Since x, y are incomparable in $P_{\vec{m}}$, $m_x \geq y$.

contradiction

Since x, z are comparable in $P_{\vec{m}}$, $m_x < z$ or $m_z < x$ (rejected b/c $m_z < x < y \Rightarrow y, z$ are comparable in $P_{\vec{m}} \Rightarrow \{y,z\} \notin E$)

$\therefore$ $y \leq m_x < z \Rightarrow x<y<z$.

Case II) $x>y$:

Since x, y are incomparable in $P_{\vec{m}}$, $m_y \geq x$

Since y, z are incomparable in $P_{\vec{m}}$, $m_y \geq z$ and $m_z \geq y$

$\because$ $\vec{m}$ is increasing

$\therefore$ $m_x > m_y \geq z$

$\because$ x, z are comparable and $m_x > z$

$\therefore$ $m_z < x$

$\therefore$ $m_z < m_y$ ($\because$ $x \leq m_y$)

$\therefore$ $z < y$ ($\because$ $m_z \leq m_y$)

$\because$ $x > y$

$\therefore$ $x > y > z$

$\therefore$ Any path $i_1 - i_2 - \cdots - i_j$ in $G_{\mathcal{K},a}$ must be either with $i_1 < i_2 < \cdots < i_j$ or $i_1 > i_2 > \cdots > i_j$.

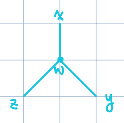
$\therefore$ $G_{\mathcal{K},a}$ has no cycle

$\because$ $G_{\mathcal{K},a}$ is bipartite and has no cycle

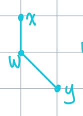
$\therefore$ $G_{\mathcal{K},a}$ is a forest

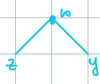
It remains to show that each vertex has degree ≤ 2.

Suppose there exists a vertex w with degree ≥ 3



$\Rightarrow$ $x<w<z$ or $x>w>z$

WLOG, assume $x<w<z$, then  with $x<w$ $\Rightarrow$ $x<w<y$

$\therefore$ $w<z$ and $w<y$, but this contradicts  ($z<w<y$ or $y<w<z$).

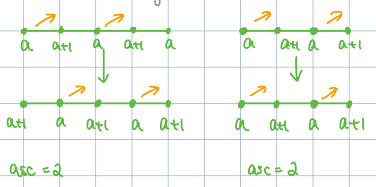
$\therefore$ All vertices have degree 0, 1 or 2.

$\therefore$ All connected component in $G_{\mathcal{K},a}$ are paths $i_1 - i_2 - \cdots - i_j$ with $i_1 < \cdots < i_j$ or $i_1 > \cdots > i_j$.

$\square$

Proof of Theorem: We define an involution on any fixed $\mathcal{K} \in K_{G_{\vec{m}}}$ preserving $\mathrm{asc}(\mathcal{K})$.
Given $a \in \mathbb{Z}_{>0}$, consider a connected component $i_1 - i_2 - \cdots - i_j$ of $G_{\mathcal{K},a}$
If j is even, do nothing (i.e. colors on $i_1 - i_2 - \cdots - i_j$ is fixed)
If j is odd, interchange a and $a+1$.

e.g.



This preserves $\mathrm{asc}(\mathcal{K})$. (b/c there are $j-1$ edge with alternating ascents and descents. If j is odd, then there are $\frac{j-1}{2}$ ascents and $\frac{j-1}{2}$ descents which will change to $\frac{j-1}{2}$ descents and $\frac{j-1}{2}$ ascents respectively. Hence asc is still $\frac{j-1}{2}$.
If j is even, there is either one more ascent or one more descent before involution. If we interchange the colors, then "one more ascent" becomes "one more descent" and vice versa. Hence we don't change the colors in that case.)

The existence of such involution proves that $X_G(x;t)$ is symmetric. $\square$