- (Weak) Stanley - Stembridge Conjecture (1990):

  $X_{G_{\vec{m}}}(x_1, x_2, \dots)$ is e-positive.

- (Strong) Stanley - Stembridge Conjecture :

  If P is 3+1 free, then $X_{\mathrm{Inc}(P)}(x_1, x_2, \dots)$ is e-positive.

- Theorem (Guay - Paquet) : Weak Stanley-Stembridge Conjecture $\Rightarrow$ Strong Stanley-Stembridge Conjecture.

- Graded Stanley - Stembridge Conjecture (Shareshian - Wachs):

  $X_{G_{\vec{m}}}(x;t)$ is e-positive.

e.g. $\vec{m} = (2,3,3)$ $P_{\vec{m}} =$  $G_{\vec{m}}$: 

$X_{G_{\vec{m}}}(x;t) = \underline{e_3} + t\,(\underline{e_3 + e_{21}}) + t^2 \underline{e_3}$

* Also, each "t-component" is e-positive.

Frobenius Characteristics (c.f. GTM 203 The Symmetric Group)

- $CF_n = \{\text{Class functions } \chi : S_n \to \mathbb{Q}\}$
- Cl_λ = conjugacy class in S_n corresponding to λ $(n = |\lambda|)$ $\left(|Cl_\lambda| = \dfrac{n!}{\prod_{i \in \mathbb{Z}_{>0}} i^{m_i} \cdot m_i!},\ m_i = \#\, i\text{'s in } \lambda \right)$

  $= \{\omega \in S_n : \omega \text{ has cycle type } \lambda\}$
- 1_{Cl_λ} = indicator function for Cl_λ $(\therefore 1_{Cl_\lambda} \in CF_n)$ (Every class function in CF_n is a linear combination of $\{1_{Cl_\lambda} : \lambda \vdash n\}$ over $\mathbb{Q}$)

e.g. $Cl_{(3)} = \{(1\,2\,3), (1\,3\,2)\}$

$Cl_{(2,1)} = \{(1\,2)(3), (1\,3)(2), (1)(2\,3)\}$

$Cl_{(1,1,1)} = \{(1)(2)(3)\} = \{id_{S_3}\}$

cycle notation. e.g. $(1\,2\,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$(1)(2\,3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$1_{Cl_{(2,1)}} : S_3 \longrightarrow \mathbb{Q}$

$\sigma \mapsto \begin{cases} 1 & \text{if } \sigma \in Cl_{(2,1)} \\ 0 & \text{otherwise} \end{cases}$

$\therefore 1_{Cl_{(2,1)}}((1\,2)(3)) = 1_{Cl_{(2,1)}}((1\,3)(2)) = 1_{Cl_{(2,1)}}((1)(2\,3)) = 1$

$1_{Cl_{(2,1)}}(id_{S_3}) = 1_{Cl_{(2,1)}}((1\,2\,3)) = 1_{Cl_{(2,1)}}((1\,3\,2)) = 0$

Frobenius characteristic : $ch : CF_n \longrightarrow \Lambda^n$ is a linear map s.t.

$1_{Cl_\lambda} \mapsto \dfrac{p_\lambda}{z_\lambda}$ $\forall \lambda \vdash n$

where $z_\lambda = |C_{S_n}(\omega)| = \dfrac{|S_n|}{|Cl_\lambda|} = \prod_{i \in \mathbb{Z}_{>0}} i^{m_i} \cdot m_i!$

e.g. $n = 3$: $ch : CF_3 \longrightarrow \Lambda^3$

$1_{Cl_{(3)}} \mapsto \dfrac{p_3}{z_3} = \dfrac{p_3}{3^1 \cdot 1!} = \dfrac{1}{3} p_3$

$1_{Cl_{(2,1)}} \mapsto \dfrac{p_{21}}{z_{21}} = \dfrac{p_{21}}{2^1 \cdot 1! \cdot 1^1 \cdot 1!} = \dfrac{1}{2} p_{21}$

$1_{Cl_{(1,1,1)}} \mapsto \dfrac{p_{111}}{z_{111}} = \dfrac{p_{111}}{1^3 \cdot 3!} = \dfrac{1}{6} p_{111}$

- $ch^{-1}(s_\lambda) = \chi^\lambda$ (irreducible character)

e.g. $s_3 = \sum_{\lambda \vdash 3} \dfrac{1}{z_\lambda} p_\lambda = \dfrac{1}{3} p_3 + \dfrac{1}{2} p_{21} + \dfrac{1}{6} p_{111}$

$ch^{-1}(s_3) = 1_{Cl_{(3)}} + 1_{Cl_{(2,1)}} + 1_{Cl_{(1,1,1)}} = \chi^{(3)}$

$s_{21} = -\dfrac{1}{3} p_3 + \dfrac{1}{3} p_{111} = (-1)\,\dfrac{1}{3} p_3 + 0 \cdot \dfrac{1}{2} p_{21} + 2\left(\dfrac{1}{6} p_{111}\right)$

$ch^{-1}(s_{21}) = (-1) \cdot 1_{Cl_{(3)}} + 0 \cdot 1_{Cl_{(2,1)}} + 2 \cdot 1_{Cl_{(1,1,1)}} = \chi^{(21)}$

$s_{111} = \sum_{\lambda \vdash 3} \dfrac{(-1)^{3-\ell(\lambda)}}{z_\lambda} p_\lambda = \dfrac{p_3}{z_3} - \dfrac{p_{21}}{z_{21}} + \dfrac{p_{111}}{z_{111}} \Rightarrow ch^{-1}(s_{111}) = 1 \cdot 1_{Cl_{(3)}} - 1 \cdot 1_{Cl_{(2,1)}} + 1 \cdot 1_{Cl_{(1,1,1)}} = \chi^{(111)}$

- $ch^{-1}(h_\lambda)$ = Perm. character (counting # fixed points on the coset $S_{\lambda_1}\times\cdots\times S_{\lambda_{\ell(\lambda)}}$)

e.g. • $h_3 = s_3$

$\therefore\ ch^{-1}(h_3) = 1_{Cl(3)} + 1_{Cl(2,1)} + 1_{Cl(1,1,1)}$

fixed points on S_3 = 1 for each $\sigma \in S_3$ b/c $\sigma S_3 = S_3$ $\forall \sigma \in S_3$

• $h_{21} = \frac{1}{2}p_{21} + \frac{1}{2}p_{111} = 1\cdot\frac{1}{2}p_{21} + 3\cdot\frac{1}{6}p_{111}$

$ch^{-1}(h_{21}) = 1_{Cl(2,1)} + 3\,1_{Cl(1,1,1)}$

(123), (132) does not fix $\langle(12)\rangle$, $\langle(13)\rangle$ or $\langle(23)\rangle$ $\quad\therefore\ 0\cdot 1_{Cl(3)}$

(12)(3), (13)(2), (23)(1) fixes exactly one of $\langle(12)\rangle$, $\langle(13)\rangle$ and $\langle(23)\rangle$ $\quad\therefore\ 1\cdot 1_{Cl(2,1)}$

(1)(2)(3) fixes all three of $\langle(12)\rangle$, $\langle(13)\rangle$, $\langle(23)\rangle$ $\quad\therefore\ 3\cdot 1_{Cl(1,1,1)}$

• $h_{111} = p_{111} = 6\cdot\frac{1}{6}p_{111}$

$ch^{-1}(h_{111}) = 6\,1_{Cl(1,1,1)}$

(123), (132), (12)(3), (13)(2), (23)(1) does not fix $\langle(1)\rangle\times\langle(2)\rangle\times\langle(3)\rangle$, $\langle(1)\rangle\times\langle(3)\rangle\times\langle(2)\rangle$, ..., $\langle(3)\rangle\times\langle(2)\rangle\times\langle(1)\rangle$

$\therefore\ 0\cdot 1_{Cl(3)} + 0\cdot 1_{Cl(2,1)}$

id_{S_3} fixes all 6 of $\langle(1)\rangle\times\langle(2)\rangle\times\langle(3)\rangle$, ..., $\langle(3)\rangle\times\langle(2)\rangle\times\langle(1)\rangle$ $\Rightarrow 6\cdot 1_{Cl(1,1,1)}$

- $ch^{-1}(e_\lambda) = ch^{-1}(h_\lambda)\otimes sgn$ $\qquad sgn(\sigma) = -1$ for $\sigma\in Cl_{(2,1)}$

e.g. $e_3 = s_{111}$ $\therefore\ ch^{-1}(e_3) = 1\cdot 1_{Cl(3)} - 1\cdot 1_{Cl(2,1)} + 1\cdot 1_{Cl(1,1,1)}$

$e_{21} = -\frac{1}{2}p_{21} + \frac{1}{2}p_{111}$

$\therefore\ ch^{-1}(e_{21}) = -1_{Cl(2,1)} + 3\cdot 1_{Cl(1,1,1)}$

$e_{111} = h_{111} = 6\cdot\frac{1}{6}p_{111}$

$\therefore\ ch^{-1}(e_{111}) = 6\cdot 1_{Cl(1,1,1)}$

Hall's inner product $\qquad$ Inner product on CF_n

- ch is an isometry, i.e. $\langle f, g\rangle = \frac{1}{n!}\sum_{\omega\in S_n} ch^{-1}(f)(\omega)\cdot ch^{-1}(g)(\omega)$

(Recall: Hall's inner product: $\langle s_\lambda, s_\mu\rangle = \langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$)

- $ch(\tau\otimes sgn) = \omega(ch(\tau))$ where $\omega(e_\lambda) = h_\lambda$

Immanants:

- $A = (a_{ij})_{i,j\in[n]}$ ($n\times n$ matrix)
- $\tau \in CF_n$

Define $Imm_\tau(A) := \sum_{\omega\in S_n} \tau(\omega)\prod_{i=1}^{n} a_{i\omega_i}$

- $\tau = sgn = \chi^{(1,1,\dots,1)}$: $Imm_\tau(A) = \det A$
- $\tau = \chi^{(n)}$ ($\therefore\ \tau(\omega)\equiv 1$) : $Imm_\tau(A) = \text{permanent}(A) = \sum_{\omega\in S_n}\prod_{i=1}^{n} a_{i\omega_i}$
- $\tau = \chi^{(2,1)}$: $Imm_\tau(A) = \underbrace{(-1)\,a_{12}a_{23}a_{31}}_{\substack{\tau((123))=-1 \\ \text{in } Cl(3)}} + \underbrace{(-1)\,a_{13}a_{21}a_{32}}_{\substack{\tau((132))=-1 \\ \text{in } Cl(3)}} + \underbrace{2\,a_{11}a_{22}a_{33}}_{\substack{\tau((1)(2)(3))=2 \\ \text{in } Cl(1,1,1)}}$ as $\chi^{(2,1)} = -1_{Cl(3)} + 2\,1_{Cl(1,1,1)}$

Jacobi-Trudi matrices:

Given $\nu, \mu \in \mathrm{Par}$ s.t. $\nu \subseteq \mu$ with $\ell(\mu) \le n$ and $|\mu/\nu| = N$, define

$$H(\mu/\nu) = (h_{\mu_i - \nu_j + j - i})_{i,j=1}^{n}$$ ← Jacobi-Trudi matrix

e.g. $n=3$, $\mu=(3,2,0)$, $\nu=(1,0,0)$

$\mu/\nu =$  $N = |\mu/\nu| = 4$

$$H(\mu/\nu) = \begin{bmatrix} h_{3-1+0} & h_{3-0+1} & h_{3-0+2} \\ h_{2-1+(-1)} & h_{2-0+0} & h_{2-0+1} \\ h_{0-1+(-2)} & h_{0-0+(-1)} & h_{0-0+0} \end{bmatrix} = \begin{bmatrix} h_2 & h_4 & h_5 \\ h_0 & h_2 & h_3 \\ h_{-3} & h_{-1} & h_0 \end{bmatrix} = \begin{bmatrix} h_2 & h_4 & h_5 \\ 1 & h_2 & h_3 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobi-Trudi identity: $s_{\mu/\nu} = \mathrm{Imm}_{\chi^{(1,1,\dots,1)}} H(\mu/\nu) = \det(H(\mu/\nu))$.

* $\deg \mathrm{Imm}_\tau (H(\mu/\nu)) = N \quad \forall \tau \in S_N$

Theorem (Gessel) (conjectured by Goulden-Jackson):

$\mathrm{Imm}_{\chi^\lambda}(H(\mu/\nu))$ is m-positive.

Theorem (Haiman) (Conjectured by Stembridge):

$\mathrm{Imm}_{\chi^\lambda}(H(\mu/\nu))$ is s-positive. (Proved by Hecke-algebra)

↑ but not e-positive (by Jacobi-Trudi)

Set $\varphi^\lambda = ch^{-1}(m_\lambda)$

Conjecture (Stembridge):

$\mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu))$ is m-positive.

(Stronger) Conjecture (Stembridge):

$\mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu))$ is s-positive.

e.g. $n=3$, $\mu=(3,2,0)$, $\nu=(1,0,0)$, $N=4$

$$H(\mu/\nu) = \begin{bmatrix} h_2 & h_4 & h_5 \\ 1 & h_2 & h_3 \\ 0 & 0 & 1 \end{bmatrix}$$ ← only consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = id$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3)$

$m_{21} = s_{21} - 2s_{111} \Rightarrow ch^{-1}(m_{21}) = \chi^{(2,1)} - 2\chi^{(1,1,1)} = -3\,1_{C\ell_{(3)}} + 2\,1_{C\ell_{(2,1)}}$

$\therefore \varphi^{(21)} = -3\,1_{C\ell_{(3)}} + 2\,1_{C\ell_{(21)}}$

$\mathrm{Imm}_{\varphi^{(21)}}(H(\mu/\nu)) = 0 \cdot h_2 h_2 \cdot 1 + 2 \cdot h_4 \cdot 1 \cdot 1 = 2h_4 = 2s_4$ which is s-positive.

Set $\delta = (n-1, n-2, \dots, 1, 0)$.

Observation: $\prod_{i=1}^{n} H(\mu/\nu)_{i\omega_i} = h_{\mu+\delta-\omega(\nu+\delta)}$

e.g. $n=3$, $\mu=(3,2,0)$, $\nu=(1,0,0)$, $\omega = id$. $\delta=(2,1,0)$

$\prod_{i=1}^{3} H(\mu/\nu)_{i\omega_i} = h_{22}$; $h_{(3,2,0)+(2,1,0)-\omega((1,0,0)+(2,1,0))} = h_{(5,3,0)-(3,1,0)} = h_{(2,2,0)} = h_{22}$

e.g. $n=3$, $\mu=(320)$, $\nu=(100)$, $\omega=(12)(3)$

$$\prod_{i=1}^{3} H(\mu/\nu)_{i\omega_i} = h_4 \;;\; h_{(3,2,0)+(2,1,0)-\omega((1,0,0)+(2,1,0))} = h_{(530)-(130)} = h_{(4,0,0)} = h_4$$

$$\therefore \ \mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu)) = \sum_{\omega\in S_n} \varphi^\lambda(\omega)\cdot\prod_{i=1}^{n} H(\mu/\nu)_{i\omega_i} = \sum_{\omega\in S_n}\varphi^\lambda(\omega)\, h_{\mu+\delta-\omega(\nu+\delta)} = \sum_{\omega\in S_n}\varphi^\lambda(\omega)\Big(\sum_{\theta\vdash N} \underbrace{K_{\theta,\mu+\delta-\omega(\nu+\delta)}}_{\text{Kostka numbers}} s_\theta\Big).$$

s- expansion of $h_{\mu+\delta-\omega(\nu+\delta)}$

e.g. $n=3$, $\mu=(320)$, $\nu=(100)$

$$\mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu)) = \varphi^\lambda(id)\, h_{22} + \varphi^\lambda((12)(3))\, h_4 = \varphi^\lambda(id)\,(s_{22}+s_{31}+s_4) + \varphi^\lambda((12)(3))\, s_4$$ where $\varphi^\lambda = ch^{-1}(m_\lambda)$, $\lambda\vdash 3$

Given $\theta\vdash N$, define $\Gamma^\theta_{\mu,\nu}(\omega) := |C_{S_n}(\omega)| \sum_{x\in cl(\omega)} K_{\theta,\mu+\delta-x(\nu+\delta)} = z_\gamma \underbrace{\sum_{x\in cl_\gamma} K_{\theta,\mu+\delta-x(\nu+\delta)}}_{\text{only depends on }\gamma}$ where γ is the cycle type of ω.

$\therefore\ \Gamma^\theta_{\mu\nu} \in CF_N$.

Then
$$\begin{aligned}
\langle \mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu)), s_\theta\rangle &= \sum_{\omega\in S_n}\varphi^\lambda(\omega)\cdot K_{\theta,\mu+\delta-\omega(\nu+\delta)}\\
&= \sum_{\omega\in S_n}\varphi^\lambda(\omega)\cdot\frac{1}{|cl(\omega)|}\sum_{x\in cl(\omega)} K_{\theta,\mu+\delta-x(\nu+\delta)}\\
&= \sum_{\omega\in S_n}\varphi^\lambda(\omega)\cdot\underbrace{\frac{1}{|cl(\omega)|}\cdot\frac{1}{|C_{S_n}(\omega)|}}_{n!}\cdot\underbrace{|C_{S_n}(\omega)|\sum_{x\in cl(\omega)} K_{\theta,\mu+\delta-x(\nu+\delta)}}_{\Gamma^\theta_{\mu,\nu}(\omega)}\\
&= \frac{1}{n!}\sum_{\omega\in S_n}\varphi^\lambda(\omega)\,\Gamma^\theta_{\mu,\nu}(\omega)\\
&= \frac{1}{n!}\sum_{\omega\in S_n} ch^{-1}(m_\lambda)(\omega)\ \ ch^{-1}(ch(\Gamma^\theta_{\mu,\nu}))(\omega)\\
&= \langle m_\lambda, ch(\Gamma^\theta_{\mu,\nu})\rangle
\end{aligned}$$

b/c each $x\in cl(\omega)$ appears $|cl(\omega)|$ times as there are $|cl(\omega)|$ choices of "ω".

$\therefore\ \mathrm{Imm}_{\varphi^\lambda}(H(\mu/\nu))$ is s- positive iff $ch(\Gamma^\theta_{\mu,\nu})$ is h- positive.

Prop. (Stanley - Stembridge): (deal with the case $\theta=(N)$)

Given μ/ν and n, there is some Hessenberg vector $\vec{m}$ s.t

$$ch(\Gamma^{(N)}_{\mu\nu}) = \omega\, X_{G_{\vec m}}(x_1,x_2,\dots)$$

Then (weak) Stanley- Stembridge Conjecture $\Rightarrow$ $ch(\Gamma^{(N)}_{\mu\nu})$ is h- positive.

$X_{G_{\vec m}}$ is e-positive

Conjecture (Stanley - Stembridge):

Given $\theta\vdash N$ and $\nu\subseteq\mu$, there exists $\mu^{(1)}/\nu^{(1)}, \mu^{(2)}/\nu^{(2)}, \dots, \mu^{(k)}/\nu^{(k)}$ s.t.

$$\Gamma^\theta_{\mu\nu} = \sum_{j=1}^{n} \Gamma^{(N)}_{\mu^{(j)}/\nu^{(j)}}.$$

* Known when θ is a hook. Nate Lesnevich has partial results on some specific shapes of θ.