Theorem: Let $\vec{m}$ be a Hessenberg vector. Then $X_{G_{\vec{m}}}(x;t)$ is Schur-positive. (See the Schur expansion at the bottom of this page)

(This gives support to the e-positivity conjecture of $X_{G_{\vec{m}}}(x;t)$ as e-positive ⇒ s-positive)

* Proved by Gasharov for (3+1)-free posets when $t=1$ (1999)) ($G_{\vec{m}}$ is (3+1)-free (and (2+2)-free))

Given a poset P on $[n]$ and $\lambda \vdash n$, a P-tableau of shape λ is a filling of the Young diagram of shape λ with $1,2,\dots,n$ (each used exactly once) such that :

(i) rows are P- increasing

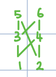
(ii) Columns have no P-decrease

$1 <_{P_{\vec{m}}} 3,4,5,6$ b/c $2<3,4,5,6$

$2 <_{P_{\vec{m}}} 4,5,6$ b/c $3<4,5,6$

e.g. $\vec{m} = (2,3,4,5,6,6)$ (n=6)

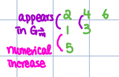
$P_{\vec{m}} =$



$G_{\vec{m}} = 1 — 2 — 3 — 4 — 5 — 6$

Choose λ, say $\lambda = (3,2,1)$ Then $\begin{array}{lll} 2 & 4 & 6 \\ 1 & 3 & \\ 5 & & \end{array}$ is a P-tableau because • rows are P-increasing $\begin{array}{lll} 2 & 4 & 6 \\ 1 & 3 & \\ 5 & & \end{array}$

• Columns are either numerical increase or unrelated in $P_{\vec{m}}$ (i.e. appears in $G_{\vec{m}}$)

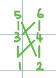


An inversion in a P-tableau T is a pair (i,j) s.t.

- $i<j$
- $\{i,j\}$ is an edge in $G_{\vec{m}}$ (i.e. i,j are incomparable in $P_{\vec{m}}$)
- i appears in a lower row than j in T

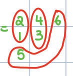
e.g. $\vec{m} = (2,3,4,5,6,6)$ (n=6)

$P_{\vec{m}} =$



$G_{\vec{m}} = 1 — 2 — 3 — 4 — 5 — 6$

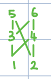
$\lambda = (3,2,1)$ $T =$



$\text{inv}_{G_{\vec{m}}}(T) = 3$ $G_{\vec{m}} =$

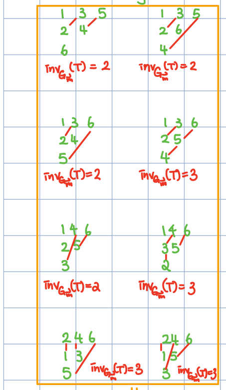
Define $\Upsilon_{P,\lambda} := \{ T : T \text{ is a P-tableau of shape } \lambda \}$

e.g. $P =$



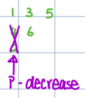
$\lambda = (3,2,1)$

Fix 1st row: (increasing in P) ⇒ 135, 136, 146, 246

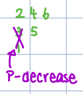


||

$\Upsilon_{P,(3,2,1)}$



(cannot have 45 as the 2nd row b/c 45 is not comparable in P)



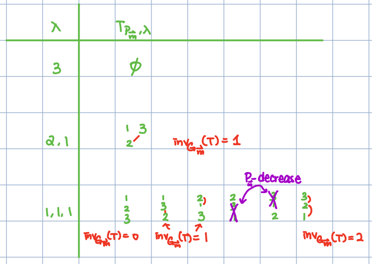
Theorem: If $\vec{m}$ is a Hessenberg vector, then $X_{G_{\vec{m}}}(x;t) = \sum_{\lambda \vdash n} s_\lambda \left(\sum_{T \in \Upsilon_{P_{\vec{m}},\lambda}} t^{\text{inv}_{G_{\vec{m}}}(T)} \right)$ and hence s-positive.

e.g. $\vec{m} = (2,3,4,5,6,6)$, $n=6$. Coeff of s_{321} in $X_{G_{\vec{m}}}(x;t) = 4t^2 + 4t^3$ and there is no s_λ with $\lambda_1 \geq 4$.

e.g. $\vec{m} = (2,3,3)$ $n=3$.

$P_{\vec{m}}$: 

$G_{\vec{m}} = 1 — 2 — 3$



$$X_{G_{(2,3,3)}}(x;t) = ts_{21} + (1+2t+t^2)s_{111} = s_{111} + t(s_{21}+2s_{111}) + t^2 s_{111}$$

Recall : $X_{G_{\vec{m}}}(x;t) := \sum_{\kappa \in K_{G_{\vec{m}}}} x^{\kappa} t^{\mathrm{asc}(\kappa)}$

There is an equivalent definition :

$$X_{G_{\vec{m}}}(x;t) := \sum_{\kappa \in K_{G_{\vec{m}}}} x^{\kappa} t^{\mathrm{des}(\kappa)}$$

just order the colors in negative
or use the involution color $i \leftrightarrow$ color $n-i+1$, n = "highest" color used
(b/c $X_{G_{\vec{m}}}$ is symmetric)

Some definitions from Gasharov:

Given a weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of n, a $P_{\vec{m}}$-array of shape α is a set partition $\pi_1 | \pi_2 | \cdots | \pi_n$ (write in increasing order) of $[n]$ such that

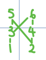
- $|\pi_i| = \alpha_i$, and
- π_i is a chain in $P_{\vec{m}}$

* Every P-tableau is a P-array.

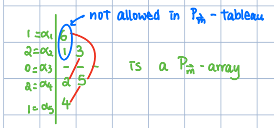
* weaker than $P_{\vec{m}}$-tableau b/c
- shape may not be a partition
- NO condition on columns

* Define inversion as usual.

e.g. $\vec{m} = (2,3,4,5,6,6)$ $\alpha = (1,2,0,2,1) \models 6$

$P_{\vec{m}} =$ 

$G_{\vec{m}} = 1 — 2 — 3 — 4 — 5 — 6$



is a $P_{\vec{m}}$ - array

Inv = 3

Write $X_{G_{\vec{m}}}(x;t) = \sum_{\lambda \vdash n} c_\lambda(t) s_\lambda$. We want to compute c_λ for each $\lambda \vdash n$.

Given $\omega \in S_n$, for $\lambda \vdash n$, define

$$\omega(\lambda) = \left(\lambda_{\omega(i)} - \omega(i) + i \right)_{i=1}^{n}$$

We only consider ω s.t. $\omega(\lambda)$ is a weak composition.

e.g. $n = 3$

ω	$\omega(2,1,0)$	
id	(2, 1, 0)	
1 3 2	(2 0 1) − (1 3 2) + (1 2 3) = (2, −1, 2)	X
2 1 3	(1 2 0) − (2 1 3) + (1 2 3) = (0, 3, 0)	
2 3 1	(1, 0, 2) − (2 3 1) + (1 2 3) = (0, −1, 4)	X
3 1 2	(0, 2, 1) − (3 1 2) + (1 2 3) = (−2, 3, 2)	X
3 2 1	(0, 1, 2) − (3 2 1) + (1 2 3) = (−2, 1, 4)	X

only consider these two

one-line notation
e.g. id $= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$
$132 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

Define $A_{P_{\vec{m}},\lambda} := \{(A,\omega) : \omega \in S_n,\ \omega(\lambda) \vDash n,\ A \text{ is a } P_{\vec{m}}\text{-array of shape } \omega(\lambda)\}$

$\uparrow$ $n =$ largest entry of $\vec{m} = |\lambda|$

e.g. $\lambda = (1,1,1)$, $\vec{m} = (2,3,3)$ $P_{\vec{m}} = \begin{smallmatrix}3\\|\\1\end{smallmatrix}\ 2$ $G_{\vec{m}} = 1 - 2 - 3$ $(n=3)$

ω	$\omega(1,1,1)$	(A,ω)
id	$(1\ 1\ 1)$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ inv=0 $\begin{smallmatrix}1\\3\\2\end{smallmatrix}$ inv=1 $\begin{smallmatrix}2\\1\\3\end{smallmatrix}$ inv=1 $\begin{smallmatrix}2\\3\\1\end{smallmatrix}$ inv=1 $\begin{smallmatrix}3\\1\\2\end{smallmatrix}$ inv=1 $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$ inv= 2 (all rows are "chains")
1 3 2	$(1\ 1\ 1) - (1\ 3\ 2) + (1\ 2\ 3) = (1\ 0\ 2)$	$\begin{smallmatrix}2\\-\ -\\1\ 3\end{smallmatrix}$ inv=1
2 1 3	$(1\ 1\ 1) - (2\ 1\ 3) + (1\ 2\ 3) = (0\ 2\ 1)$	$\begin{smallmatrix}-\ -\\1\ 3\\2\end{smallmatrix}$ inv = 1
2 3 1	$(1\ 1\ 1) - (2\ 3\ 1) + (1\ 2\ 3) = (0\ 0\ 3)$	ϕ (no chain of length ≥ 3 in $P_{\vec{m}}$)
3 1 2	$(1\ 1\ 1) - (3\ 1\ 2) + (1\ 2\ 3) = (①\ 2\ 2)$ ✗	
3 2 1	$(1\ 1\ 1) - (3\ 2\ 1) + (1\ 2\ 3) = (①\ 1\ 3)$ ✗	

Claim: $c_\lambda(t) = \sum_{(A,\omega)\in A_{P_{\vec{m}},\lambda}} \operatorname{sgn}(\omega)\, t^{\operatorname{inv}_{G_{\vec{m}}}(A)}$

e.g. $\vec{m} = (2,3,3)$, $\lambda = (1,1,1)$

$c_{111}(t) = \operatorname{sgn}(\mathrm{id})\cdot(t^0 + 4t + t^2) + \operatorname{sgn}(132)\cdot t + \operatorname{sgn}(213)\cdot t = 1 + 4t + t^2 - t - t = 1 + 2t + t^2$

$\uparrow$

coeff of s_{111} in $X_{G_{(2,3,3)}}(x,t)$

* Every array corresponds to a proper coloring with colors $1,2,\dots,n$: Elements in row i gets color i (b/c elements in each row are comparable, hence none of them form an edge in $G_{\vec{m}}$.)

How does the claim above imply the theorem?

- If $A \in \mathcal{T}_{P_{\vec{m}},\lambda}$, i.e. if A is a $P_{\vec{m}}$-tableau of shape λ, then $(A, \mathrm{id}) \in A_{P_{\vec{m}},\lambda}$ (b/c $P_{\vec{m}}$-Tableau is also a $P_{\vec{m}}$-array)
- If $\omega \neq \mathrm{id}$, then $\omega(\lambda)$ is not a partition ($\lambda \in \mathrm{Par}$). Hence if $(A,\omega) \in A_{P_{\vec{m}},\lambda}$, then A's shape is not a partition. i.e. $A \notin \mathcal{T}_{P_{\vec{m}},\lambda}$ (i.e. $A \in \mathcal{T}_{P_{\vec{m}},\lambda}$ and $\omega \neq \mathrm{id}$ $\Rightarrow (A,\omega) \notin A_{P_{\vec{m}},\lambda}$)

$\because\ \omega \neq \mathrm{id}$

$\therefore\ \exists\ i<j$ s.t. $\omega(i) > \omega(j)$

$\therefore\ \left.\begin{array}{l}\lambda_{\omega(i)} \leq \lambda_{\omega(j)}\\ -\omega(i) < -\omega(j)\\ i < j\end{array}\right\} \Rightarrow \lambda_{\omega(i)} - \omega(i) + i < \lambda_{\omega(j)} - \omega(j) + j \Rightarrow (\omega(\lambda))_i < (\omega(\lambda))_j \Rightarrow \omega(\lambda) \notin \mathrm{Par}$

$\therefore$ For $A \in \mathcal{T}_{P_{\vec{m}},\lambda}$, $(A,\omega) \in A_{P_{\vec{m}},\lambda}$ iff $\omega = \mathrm{id}$.

Define $\mathcal{B}_{P_{\vec{m}},\lambda} := A_{P_{\vec{m}},\lambda} \setminus \{(A,\mathrm{id}) : A \in \mathcal{T}_{P_{\vec{m}},\lambda}\}$

e.g. $\lambda = (1,1,1)$, $\vec{m} = (2,3,3)$ $P_{\vec{m}} = \begin{smallmatrix}3\\|\\1\end{smallmatrix}\ 2$ $G_{\vec{m}} = 1 - 2 - 3$ $(n=3)$

ω	$\omega(1,1,1)$	(A,ω)
id	$(1\ 1\ 1)$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$ $\begin{smallmatrix}1\\3\\2\end{smallmatrix}$ $\begin{smallmatrix}2\\1\\3\end{smallmatrix}$ $\boxed{\begin{smallmatrix}2\\3\\1\end{smallmatrix}}$ $\boxed{\begin{smallmatrix}3\\1\\2\end{smallmatrix}}$ $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$
1 3 2	$(1\ 1\ 1) - (1\ 3\ 2) + (1\ 2\ 3) = (1\ 0\ 2)$	$\boxed{\begin{smallmatrix}2\\-\ -\\1\ 3\end{smallmatrix}}$
2 1 3	$(1\ 1\ 1) - (2\ 1\ 3) + (1\ 2\ 3) = (0\ 2\ 1)$	$\boxed{\begin{smallmatrix}-\ -\\1\ 3\\2\end{smallmatrix}}$
2 3 1	$(1\ 1\ 1) - (2\ 3\ 1) + (1\ 2\ 3) = (0\ 0\ 3)$	ϕ
3 1 2	$(1\ 1\ 1) - (3\ 1\ 2) + (1\ 2\ 3) = (①\ 2\ 2)$ ✗	
3 2 1	$(1\ 1\ 1) - (3\ 2\ 1) + (1\ 2\ 3) = (①\ 1\ 3)$ ✗	

These form $\mathcal{B}_{P_{(233)},(111)}$.

what we need to prove the theorem

If we can find a sign-reversing and $\operatorname{inv}_{G_{\vec{m}}}$-preserving involution on $\mathcal{B}_{P_{\vec{m}},\lambda}$, then

$$c_\lambda(t) = \sum_{(A,\omega)\in A_{P_{\vec{m}},\lambda}} \operatorname{sgn}(\omega)\, t^{\operatorname{inv}_{G_{\vec{m}}}(A)} = \overbrace{\sum_{(A,\omega)\in \mathcal{B}_{P_{\vec{m}},\lambda}} \operatorname{sgn}(\omega)\, t^{\operatorname{inv}_{G_{\vec{m}}}(A)}}^{0} + \sum_{A\in\mathcal{T}_{P_{\vec{m}},\lambda}} \operatorname{sgn}(\mathrm{id})\, t^{\operatorname{inv}_{G_{\vec{m}}}(A)} = \underset{\downarrow}{\sum_{A\in\mathcal{T}_{P_{\vec{m}},\lambda}} t^{\operatorname{inv}_{G_{\vec{m}}}(A)}}$$

Given $(A,\omega) \in \mathcal{B}_{P_{\vec{m}},\lambda}$, call an entry x of A "bad" if either : (i) There is no entry of A directly above x when x is not in the top row, or
(ii) The entry y directly above x satisfies $y >_{P_{\vec{m}}} x$.

} These are the conditions which make $A \notin \mathcal{T}_{P_{\vec{m}},\lambda}$ (i.e. A not a tableau)

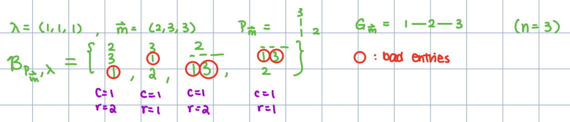
e.g. $\lambda = (1,1,1)$, $\vec{m} = (2,3,3)$ $P_{\vec{m}} = \begin{smallmatrix}3\\|\\1\end{smallmatrix}\ 2$ $G_{\vec{m}} = 1 - 2 - 3$ $(n=3)$

$\mathcal{B}_{P_{\vec{m}},\lambda} = \left\{ \begin{smallmatrix}2\\3\\①\end{smallmatrix},\ \begin{smallmatrix}3\\①\\2\end{smallmatrix},\ \begin{smallmatrix}2\\-\ -\\①\ ③\end{smallmatrix},\ \begin{smallmatrix}-\ -\\①\ ③\\2\end{smallmatrix} \right\}$

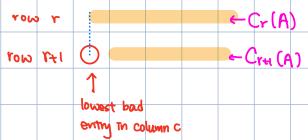
○ : bad entries

Find the leftmost column c in A containing a bad entry, and let r be the row above the lowest bad entry in column c.

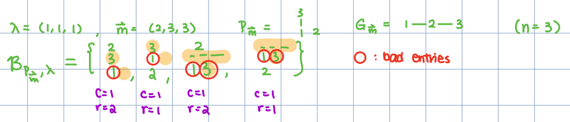
e.g. $\lambda=(1,1,1)$, $\vec{m}=(2,3,3)$



Let $C_r(A)$ be the set of entries in row r weakly to the right of column c.
Let $C_{r+1}(A)$ be the set of entries in row r+1 strictly to the right of column c.



e.g. $\lambda=(1,1,1)$, $\vec{m}=(2,3,3)$



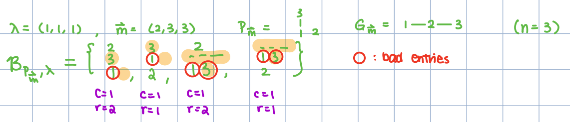
$C_2(A)=\{3\}$	$C_1(A)=\{3\}$	$C_2(A)=\phi$	$C_1(A)=\phi$
$C_3(A)=\phi$	$C_2(A)=\phi$	$C_3(A)=\{3\}$	$C_2(A)=\{3\}$

Let $H(A)$ = subgraph of $G_{\vec{m}}$ induced on $C_r(A)\cup C_{r+1}(A)$ (bipartite because entries in the same row in A are comparable, hence not an edge in $G_{\vec{m}}$)

For $i\in\{r,r+1\}$, define :

- $O_i(A) := \{x\in C_i(A) \mid x$ is in a connected component of odd size in $H(A)\}$
- $E_i(A) := C_i(A)\setminus O_i(A)$
- $I_i(A) := \{$entries in row i but not in $C_i(A)\}$

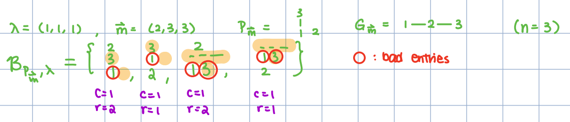
e.g. $\lambda=(1,1,1)$, $\vec{m}=(2,3,3)$



$C_2(A)=\{3\}$	$C_1(A)=\{3\}$	$C_2(A)=\phi$	$C_1(A)=\phi$
$C_3(A)=\phi$	$C_2(A)=\phi$	$C_3(A)=\{3\}$	$C_2(A)=\{3\}$
$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$
$O_2(A)=\{3\}$	$O_1(A)=\{3\}$	$O_2(A)=\phi$	$O_1(A)=\phi$
$E_2(A)=\phi$	$E_1(A)=\phi$	$E_2(A)=\phi$	$E_1(A)=\phi$
$I_2(A)=\phi$	$I_1(A)=\phi$	$I_2(A)=\phi$	$I_1(A)=\phi$
$O_3(A)=\phi$	$O_2(A)=\phi$	$O_3(A)=\{3\}$	$O_2(A)=\{3\}$
$E_3(A)=\phi$	$E_2(A)=\phi$	$E_3(A)=\phi$	$E_2(A)=\phi$
$I_3(A)=\{1\}$	$I_2(A)=\{1\}$	$I_3(A)=\{1\}$	$I_2(A)=\{1\}$

Claim: We can exchange $\underbrace{O_r(A) \text{ with } O_{r+1}(A)}_{\text{array with the same inversion}}$ and replaces $\underbrace{\omega \text{ with } \omega(r\ r+1)}_{\text{sign-reversing}}$

e.g. $\lambda=(1,1,1)$, $\vec{m}=(2,3,3)$



$C_2(A)=\{3\}$	$C_1(A)=\{3\}$	$C_2(A)=\phi$	$C_1(A)=\phi$
$C_3(A)=\phi$	$C_2(A)=\phi$	$C_3(A)=\{3\}$	$C_2(A)=\{3\}$
$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$	$H(A)$: $\underset{3}{\bullet}$
$O_2(A)=\{3\}$	$O_1(A)=\{3\}$	$O_2(A)=\phi$	$O_1(A)=\phi$
$E_2(A)=\phi$	$E_1(A)=\phi$	$E_2(A)=\phi$	$E_1(A)=\phi$
$I_2(A)=\phi$	$I_1(A)=\phi$	$I_2(A)=\phi$	$I_1(A)=\phi$
$O_3(A)=\phi$	$O_2(A)=\phi$	$O_3(A)=\{3\}$	$O_2(A)=\{3\}$
$E_3(A)=\phi$	$E_2(A)=\phi$	$E_3(A)=\phi$	$E_2(A)=\phi$
$I_3(A)=\{1\}$	$I_2(A)=\{1\}$	$I_3(A)=\{1\}$	$I_2(A)=\{1\}$

$\begin{matrix}2\\3\\1\end{matrix}$ $\longleftrightarrow$

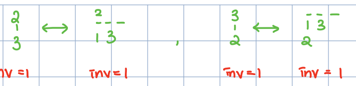
$\omega=id$
$r=2$
$O_2(A)=\{3\}$
$O_3(A)=\phi$

$\omega=(2,3)=132$ ↙ one-line notation $(\therefore \omega(111)=102)$
$O_2(A)=\phi$
$O_3(A)=\{3\}$
} which has exactly this

$\begin{matrix}3\\|\\2\end{matrix}$ $\longleftrightarrow$

$\omega=id$
$r=1$
$O_1(A)=\{3\}$
$O_2(A)=\phi$

$\omega=(12)=213$
$O_1(A)=\phi$
$O_2(A)=\{3\}$
} Which is exactly this

Thus 

To prove the claim, we need to check :

① The resulting filling is a $P_{\vec{m}}$-array of shape $\omega(r\ r{+}1)(\lambda)$.

② The map is an involution

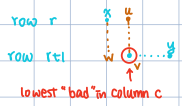
③ The map preserves $\mathrm{inv}_{G_{\vec{m}}}$.

These are all the entries in the image of the map in row r.

To prove ①: (i) We want to show that $I_r(A) \cup E_r(A) \cup O_{r+1}(A)$ is a chain in $P_{\vec{m}}$.

Since $I_r(A) \cup O_r(A)$ is part of row r in A, we only need to consider $x \in I_r(A) \cup E_r(A)$ and $y \in O_{r+1}(A)$

- If $x \in I_r(A)$ and $y \in O_{r+1}(A)$, then we want to show $x <_{P_{\vec{m}}} y$.



We know $u >_{P_{\vec{m}}} v$ b/c v is "bad", $w <_{P_{\vec{m}}} v <_{P_{\vec{m}}} y$ b/c w, v, y is a chain in $P_{\vec{m}}$

Since $P_{\vec{m}}$ is (3+1)-free, x is comparable with one of w, v, y.

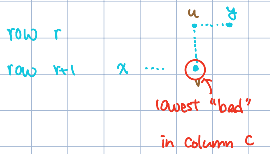
However $x \not>_{P_{\vec{m}}} w$ (otherwise c is not the leftmost column with a "bad" entry)

∴ Either $x <_{P_{\vec{m}}} w$ or $x <_{P_{\vec{m}}} v <_{P_{\vec{m}}} y$ or $x <_{P_{\vec{m}}} y$. (In any case, $x <_{P_{\vec{m}}} y$)

- If $x \in E_r(A)$ and $y \in O_{r+1}(A)$, then as x any y are in separated connected components in H(A) which is a subgraph of $G_{\vec{m}}$, x and y are comparable.

(ii) Similar to (i), we want to show $I_{r+1}(A) \cup E_{r+1}(A) \cup O_r(A)$ is a chain in $P_{\vec{m}}$.

As before, we only need to consider the case when $x \in I_{r+1}(A)$ and $y \in O_r(A)$.



We know $x <_{P_{\vec{m}}} v$ (same row) , $v <_{P_{\vec{m}}} u$, (v is "bad") $u <_{P_{\vec{m}}} y$ (same row) $\Rightarrow$ $x <_{P_{\vec{m}}} y$.

(iii) Recall in Lecture 1 last Lemma, since H(A) is an induced graph of $G_{\mathcal{K},r}$ (where $\mathcal{K}$ is the proper coloring corresponding to A), its connected components are paths. ∴ Each connected component of even size in H(A) has exactly half of its elements in row r in A.

∴ The new row r has $\underset{|I_r(A)|}{\underset{\uparrow}{c-1}} + \underbrace{\omega(\lambda)_{r+1} - c}_{|C_{r+1}(A)|} = \omega(\lambda)_{r+1} - 1$ entries.

The new row r+1 has $c + \omega(\lambda)_r - (c-1) = \omega(\lambda)_r + 1$ entries.

$$
\begin{aligned}
\because \big(\omega(r\ r{+}1)(\lambda)\big)_r &= \lambda_{\omega(r\ r+1)(r)} - \omega(r\ r{+}1)(r) + r \\
&= \lambda_{\omega(r+1)} - \omega(r{+}1) + r \\
&= \omega(\lambda)_{r+1} - 1 \\
\big(\omega(r\ r{+}1)(\lambda)\big)_{r+1} &= \lambda_{\omega(r\ r+1)(r+1)} - \omega(r\ r{+}1)(r{+}1) + r{+}1 \\
&= \lambda_{\omega(r)} - \omega(r) + r \ + 1 \\
&= \omega(\lambda)_r + 1
\end{aligned}
$$

∴ The new row r has length $\big(\omega(r\ r{+}1)(\lambda)\big)_r$ and new row r+1 has length $(\omega(r\ r{+}1)(\lambda))_{r+1}$.

∴ The new array has shape $\omega(r\ r{+}1)(\lambda)$.

To prove ②: Let A′ be the image of A under the map.

By construction, only entries strictly to the right in row r+1 and above the bad entry in row r and column c in row r+1 are affected, if we can prove that they share the same leftmost lowest bad entry, then we can prove that A and A′ has the same c, r (Then the map is valid and hence clearly an involution by construction)

A : row r u

row $r+1$ $\textcircled{v}$

↑ column c (lowest bad entry)

A' : row r $w \leftarrow \in E_r(A) \cup O_{r+1}(A) \cup \{0\}$

row $r+1$ $\textcircled{v}$

↑ Column c

If $w \in O_{r+1}(A)$, then $w >_{P_{\vec{m}}} v$ (in A)

If $w \in E_r(A)$, then $w \geq_{P_{\vec{m}}} u >_{P_{\vec{m}}} v$

$\Big\} \Rightarrow w >_{P_{\vec{m}}} v \Rightarrow v$ is again the leftmost lowest bad entry

$\therefore$ A and A' share the same r and c.

$O_r(A') = O_{r+1}(A)$, $O_{r+1}(A') = O_r(A)$

$E_r(A') = E_r(A)$, $E_{r+1}(A') = E_{r+1}(A)$

$\therefore (A')' = A$ ($\therefore$ involution).

an inversion in A means a descent in coloring

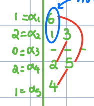
To prove ③ : To prove $\mathrm{inv}_{G_{\vec{m}}}(A) = \mathrm{inv}_{G_{\vec{m}}}(A')$, it is the same as showing $\mathrm{des}(\mathcal{K}) = \mathrm{des}(\mathcal{K}')$ where $\mathcal{K}$ and $\mathcal{K}'$ are the associate coloring of (elements in row i gets color i) A and A' respectively.

e.g. $\vec{m} = (2,3,4,5,6,6)$ $\alpha = (1,2,0,2,1) \vDash 6$

$P_{\vec{m}} =$ 

$\mathcal{K}$: 2 4 — 2 5 — 4 — 1 des = 3

$G_{\vec{m}} =$ 1 — 2 — 3 — 4 — 5 — 6



not allowed in $P_{\vec{m}}$ - tableau

is a $P_{\vec{m}}$ - array

inv = 3

Note that $\mathcal{K}'$ is obtained from $\mathcal{K}$ by replacing r by $r+1$ and $r+1$ by r in each odd size connected component in $H(A)$ (i.e. odd path)

$\therefore$ # descents in $\mathcal{K}$ = # ascents in $\mathcal{K}$ = $\frac{\#\text{edges}}{2}$ ($\because$ odd path)

$\therefore$ # ascents in $\mathcal{K}'$ = # descents in $\mathcal{K}'$ = $\frac{\#\text{edges}}{2}$

$\therefore$ # descents in $\mathcal{K}$ = # descents in $\mathcal{K}'$

$\therefore \mathrm{inv}_{G_{\vec{m}}}(A) = \mathrm{inv}_{G_{\vec{m}}}(A')$.

It remains to prove the following claim:

Claim: $c_\lambda(t) = \sum_{(A,\omega) \in A_{P_{\vec{m}},\lambda}} \mathrm{sgn}(\omega)\, t^{\mathrm{inv}_{G_{\vec{m}}}(A)}$

Proof: $c_\lambda(t) := \langle X_{G_{\vec{m}}}(x;t), s_\lambda \rangle$

$= \sum_{\omega \in S_n} \mathrm{sgn}(\omega) \langle X_{G_{\vec{m}}}(x;t), h_{\omega(\lambda)} \rangle$ (by Jacobi-Trudi)

$= \sum_{\omega \in S_n} \mathrm{sgn}(\omega) \cdot \big(\text{coeff of } m_{\omega(\lambda)} \text{ in } X_{G_{\vec{m}}}(x;t)\big)$ (b/c $\langle m_\mu, h_\lambda \rangle = \delta_{\mu\lambda}$)

$= \sum_{\omega \in S_n} \mathrm{sgn}(\omega) \cdot \big(\text{coeff of } \prod_{i=1}^{n} x^{\omega(\lambda)_i} \text{ in } X_{G_{\vec{m}}}(x;t)\big)$

// by definition

$= \sum_{\omega \in S_n} \mathrm{sgn}(\omega) \cdot \underbrace{\sum_{\mathcal{K} \in K_{\omega(\lambda)}}}_{} t^{\mathrm{des}(\mathcal{K})}$

Proper colorings satisfying

$|\mathcal{K}^{-1}(i)| = \omega(\lambda)_i$

($\therefore$ rows with size $\omega(\lambda)_i$ gets color i)

$= \sum_{(A,\omega) \in A_{P_{\vec{m}},\lambda}} \mathrm{sgn}(\omega)\, t^{\mathrm{inv}_{G_{\vec{m}}}(A)}$.