Notations: For $\sigma \in S_n$, define

n: positive integer

- $\mathrm{des}(\sigma) := |\{ i \in [n-1] : \sigma(i) > \sigma(i+1) \}|$
- $\mathrm{exc}(\sigma) := |\{ i \in [n-1] : \sigma(i) > i \}|$
- $\mathrm{Des}(\sigma) := \{ i \in [n-1] : \sigma(i) > \sigma(i+1) \}$ ($\therefore \mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|$)
- $\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i$
- $\mathrm{inv}(\sigma) = |\{ (i,j) : i<j \text{ and } \sigma(i) > \sigma(j) \}|$

$\Bigg\}$ Mahonian (i.e. $\sum_{\sigma \in S_n} t^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} = [n]_q! = \prod_{i=1}^{n} (1+q+\cdots+q^{i-1})$)

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 5 & 3 & 8 & 2 & 7 \end{pmatrix}$

$\mathrm{Des}(\sigma) = \{1, 4, 6\}$

$\mathrm{des}(\sigma) = 3$

$\mathrm{maj}(\sigma) = 1+4+6 = 11$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 5 & 3 & 8 & 2 & 7 \end{pmatrix}$

$\mathrm{exc}(\sigma) = |\{1, 3, 4, 6\}| = 4$

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 5 & 3 & 8 & 2 & 7 \end{pmatrix}$

$\mathrm{inv}(\sigma) = |\{(1,3), (1,4), (1,5), (1,7), (3,5), (3,7), (4,5), (4,7), (5,7), (6,7), (6,8)\}| = 11.$

Define the Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)}$$

(* It is more common to define $A_n(t)$ as $\sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)+1} = t\left(\sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)}\right)$)

Fact: $A_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{exc}(\sigma)}$ (i.e. $\mathrm{des}(\sigma)$ and $\mathrm{exc}(\sigma)$ are equidistributed)

e.g.

σ	$\mathrm{des}(\sigma)$	$\mathrm{exc}(\sigma)$
$\binom{1\,2\,3}{1\,2\,3}$	0	0
$\binom{1\,2\,3}{1\,3\,2}$	1	1
$\binom{1\,2\,3}{2\,1\,3}$	1	1
$\binom{1\,2\,3}{2\,3\,1}$	1	2
$\binom{1\,2\,3}{3\,1\,2}$	1	1
$\binom{1\,2\,3}{3\,2\,1}$	2	1

$$\left.\begin{aligned} \therefore \sum_{\sigma \in S_3} t^{\mathrm{des}(\sigma)} &= t^0 + 4t + t^2 = 1+4t+t^2 \\ \sum_{\sigma \in S_3} t^{\mathrm{exc}(\sigma)} &= t^0 + 4t + t^2 = 1+4t+t^2 \end{aligned}\right\} = A_3(t)$$

e.g.

σ (one-line)	$\mathrm{des}(\sigma)$	$\mathrm{exc}(\sigma)$
1234	0	0
1243	1	1
1324	1	1
1342	1	2
1423	1	1
1432	2	1
2134	1	1
2143	2	2
2314	1	2
2341	1	3
2413	1	2
2431	2	2
3124	1	1
3142	2	2
3214	2	1
3241	2	2
3412	1	2
3421	2	2
4123	1	1
4132	2	1
4213	2	1
4231	2	1
4312	2	2
4321	3	2

$$\left.\begin{aligned} \therefore \sum_{\sigma \in S_4} t^{\mathrm{des}(\sigma)} &= t^0 + 11t + 11t^2 + t^3 = 1 + 11t + 11t^2 + t^3 \\ \sum_{\sigma \in S_4} t^{\mathrm{exc}(\sigma)} &= t^0 + 11t + 11t^2 + t^3 = 1 + 11t + 11t^2 + t^3 \end{aligned}\right\} A_4(t)$$

Write $A_n(t) = \sum_{j \geq 0} \left\langle {n \atop j} \right\rangle t^j$.

$\left\langle {n \atop j} \right\rangle = \# \sigma \in S_n$ s.t. $\mathrm{des}(\sigma) = j$ (Eulerian number)

e.g. $A_3(t) = 1 + 4t + t^2$

$\left\langle {3 \atop 0} \right\rangle = 1$ $\left\langle {3 \atop 1} \right\rangle = 4$ $\left\langle {3 \atop 2} \right\rangle = 1$

$A_4(t) = 1 + 11t + 11t^2 + t^3$

$\left\langle {4 \atop 0} \right\rangle = 1$ $\left\langle {4 \atop 1} \right\rangle = 11$ $\left\langle {4 \atop 2} \right\rangle = 11$ $\left\langle {4 \atop 3} \right\rangle = 1$

Properties:

- $\left\langle {n \atop j} \right\rangle = 0$ for all $j \geq n$ (b/c $\mathrm{des}(\sigma) \leq n-1$ $\forall \sigma \in S_n$)
- $\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1$
- palindromic: $\left\langle {n \atop j} \right\rangle = \left\langle {n \atop n-1-j} \right\rangle$
- unimodal: $\left\langle {n \atop 0} \right\rangle \leq \left\langle {n \atop 1} \right\rangle \leq \cdots \leq \left\langle {n \atop \lfloor \frac{n-1}{2} \rfloor} \right\rangle \geq \left\langle {n \atop \lfloor \frac{n+1}{2} \rfloor} \right\rangle \geq \cdots \geq \left\langle {n \atop n-1} \right\rangle$

q- analogue:

$$A_n(q,t) := \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} \qquad (\text{Set } A_0(q,t) = 1)$$

$\therefore A_n(1,t) = A_n(t)$

e.g.

σ	$\mathrm{maj}(\sigma)$	$\mathrm{exc}(\sigma)$
$\begin{pmatrix} 1&2&3 \\ 1&2&3 \end{pmatrix}$	0	0
$\begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix}$	2	1
$\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}$	1	1
$\begin{pmatrix} 1&2&3 \\ 2&3&1 \end{pmatrix}$	2	2
$\begin{pmatrix} 1&2&3 \\ 3&1&2 \end{pmatrix}$	1	1
$\begin{pmatrix} 1&2&3 \\ 3&2&1 \end{pmatrix}$	1+2=3	1

$$\therefore A_3(q,t) = q^{0-0} t^0 + q^{2-1} t + q^{1-1} t + q^{2-2} t^2 + q^{1-1} t + q^{3-1} t = 1 + (2 + q + q^2) t + t^2$$

e.g.

σ (one-line)	$\mathrm{maj}(\sigma)$	$\mathrm{exc}(\sigma)$	$\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$
1234	0	0	0
1243	3	1	2
1324	2	1	1
1342	3	2	1
1423	2	1	1
1432	2+3=5	1	4
2134	1	1	0
2143	1+3=4	2	2
2314	2	2	0
2341	3	3	0
2413	2	2	0
2431	2+3=5	2	3
3124	1	1	0
3142	1+3=4	2	2
3214	1+2=3	1	2
3241	1+3=4	2	2
3412	2	2	0
3421	2+3=5	2	3
4123	1	1	0
4132	1+3=4	1	3
4213	1+2=3	1	2
4231	1+3=4	1	3
4312	1+2=3	2	1
4321	1+2+3=6	2	4

$$\therefore A_4(q,t) = t^0 q^0 + t(q^2 + q + q + q^4 + 1 + 1 + q^2 + 1 + q^3 + q^2 + q^3) + t^2(q + q^2 + 1 + 1 + q^3 + q^2 + q^2 + 1 + q^3 + q + q^4) + t^3$$

$$= 1 + (3 + 2q + 3q^2 + 2q^3 + q^4) t + (3 + 2q + 3q^2 + 2q^3 + q^4) t^2 + t^3$$

Notations: $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ $([0]_q := 1)$

$[n]_q! := [n]_q \cdot [n-1]_q \cdot \cdots \cdot [1]_q$

e.g. $[3]_q = 1 + q + q^2$, $[3]_q! = (1 + q + q^2)(1 + q)(1)$

Notation : $\exp_q(u) = \sum_{n\geq 0} \frac{u^n}{[n]_q!}$ (When $q=1$, this is the usual Euler's exponential generating function $\exp(u) := \sum_{n\geq 0} \frac{u^n}{n!}$.)

Theorem: (Shareshian & Wachs, 2007)

$$\sum_{n\geq 0} A_n(q,t)\frac{u^n}{[n]_q!} = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$$

This implies : (1) $A_n(q,t)$ is palindromic as a polynomial in t (coeff of t^j = coeff of t^{n-1-j} $\forall\, 0\leq j\leq n-1$)

(2) $A_n(q,t)$ is q-unimodal (meaning the coeff. of t^m − coeff of t^{m-1} is positive in q, $\forall\, 1\leq m\leq \lfloor \frac{n-1}{2} \rfloor$)

* Sketch of proof of (1) and (2): (See Appendix B.1, B.3, C.1, C.2 in Chromatic quasisymmetric functions)

- $$\sum_{n\geq 0} A_n(q,t)\frac{u^n}{[n]_q!} = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$$

i.e. $$1 + \sum_{n\geq 1} A_n(q,t)\frac{u^n}{[n]_q!} = \frac{(1-t)\sum_{n\geq 0}\frac{u^n}{[n]_q!}}{\sum_{n\geq 0}\frac{(tu)^n}{[n]_q!} - t\sum_{n\geq 0}\frac{u^n}{[n]_q!}}$$

$$= \frac{(1-t)\sum_{n\geq 0}\frac{u^n}{[n]_q!}}{\sum_{n\geq 0}(t^n - t)\frac{u^n}{[n]_q!}}$$

$$= \frac{(1-t)\sum_{n\geq 0}\frac{u^n}{[n]_q!}}{(1-t) + t\sum_{n\geq 2}(t^{n-1}-1)\frac{u^n}{[n]_q!}}$$

$$= \frac{\sum_{n\geq 0}\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}(1+t+\cdots+t^{n-2})\frac{u^n}{[n]_q!}}$$

$$= \frac{\sum_{n\geq 0}\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}[n-1]_t\frac{u^n}{[n]_q!}}$$

$$= 1 + \frac{\cancel{1} + u + \sum_{n\geq 2}\frac{u^n}{[n]_q!} \cancel{-1} + \overbrace{t\sum_{n\geq 2}[n-1]_t}^{t+t^2+\cdots+t^{n-1}}\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}[n-1]_t\frac{u^n}{[n]_q!}}$$

$$= 1 + \frac{u + \sum_{n\geq 2}(1+t+\cdots+t^{n-1})\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}[n-1]_t\frac{u^n}{[n]_q!}}$$

$$= 1 + \frac{\sum_{n\geq 1}(1+\cdots+t^{n-1})\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}[n-1]_t\frac{u^n}{[n]_q!}}$$

$$= 1 + \frac{\sum_{n\geq 1}[n]_t\frac{u^n}{[n]_q!}}{1 - t\sum_{n\geq 2}[n-1]_t\frac{u^n}{[n]_q!}}$$

$$= 1 + \left(\sum_{r\geq 1}[r]_t\frac{u^r}{[r]_q!}\right)\left(\sum_{i\geq 0}\left(t\sum_{j\geq 2}[j-1]_t\frac{u^j}{[j]_q!}\right)^i\right)$$

$$= 1 + \left(\sum_{r\geq 1}[r]_t\frac{u^r}{[r]_q!}\right)\left(\sum_{i\geq 0}t^i\cdot\sum_{m\geq 0}\sum_{\substack{a_1,\ldots,a_i\geq 2\\ a_1+\cdots+a_i=m}}[a_1-1]_t[a_2-1]_t\cdots[a_i-1]_t\cdot\frac{u^m}{[a_1]_q!\cdot\cdots\cdot[a_i]_q!}\right)$$

$$= 1 + \left(\sum_{r\geq 1}[r]_t\frac{u^r}{[r]_q!}\right)\left(\sum_{m\geq 0}\sum_{i\geq 0}t^i\cdot\sum_{\substack{a_1,\ldots,a_i\geq 2\\ a_1+\cdots+a_i=m}}\prod_{r=1}^{i}[a_r-1]_t\cdot\frac{u^m}{\prod_{k=1}^{i}[a_k]_q!}\right)$$

Choose r, m s.t. $r+m=n$

Comparing coefficients of $\frac{u^n}{[n]_q!}$ on both sides, we get

$$A_n(q,t) = \sum_{m\geq 0}\sum_{i\geq 0}t^i\cdot\sum_{\substack{a_1,\ldots,a_i\geq 2\\ a_1+\cdots+a_i=m}}\left(\prod_{r=1}^{i}[a_r-1]_t\right)\left([n-m]_t\right)\cdot\frac{[n]_q!}{[n-m]_q!\cdot\prod_{k=1}^{i}[a_k]_q!}$$

Note that $r \geq 1$, so $n-m \geq 1$ ← our choice of r

Set $a_{i+1} = n-m+1$ $\quad (\therefore a_{i+1} \geq 2$ and $[n-m]_t = [a_{i+1}-1]_t)$

$$A_n(q,t) = \sum_{i \geq 0} t^i \sum_{\substack{a_1,\dots,a_i,\, a_{i+1} \geq 2 \\ a_1+\cdots+a_{i+1} = n+1}} \left(\prod_{r=1}^{i+1} [a_r - 1]_t \right) \cdot \underbrace{\frac{[n]_q!}{[a_{i+1}-1]_q!\,[a_1]_q! \cdots [a_i]_q!}}_{\begin{bmatrix} n \\ a_1\ a_2 \cdots a_i\ \ a_{i+1}-1 \end{bmatrix}_q}$$

no m anymore

$$= \sum_{i \geq 1} t^{i-1} \sum_{\substack{a_1,\dots,a_i \geq 2 \\ a_1+\cdots+a_i = n+1}} \prod_{r=1}^{i} [a_r - 1]_t \cdot \begin{bmatrix} n \\ a_1\ a_2 \cdots a_{i-1}\ \ a_i - 1 \end{bmatrix}_q$$

b/c $a_1,\dots,a_i \geq 2$ and $a_1+\cdots+a_i = n+1$

$$= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} t^{i-1} \sum_{\substack{a_1,\dots,a_i \geq 2 \\ a_1+\cdots+a_i = n+1}} \prod_{r=1}^{i} [a_r - 1]_t \cdot \begin{bmatrix} n \\ a_1\ a_2 \cdots a_{i-1}\ \ a_i - 1 \end{bmatrix}_q$$

$\because$ $t^{i-1} \cdot \prod_{r=1}^{i} [a_r - 1]_t$ is positive, unimodal and palindromic with centre of symmetry $i-1+\sum_{r=1}^{i} \frac{a_r - 2}{2} = \frac{1}{2}\sum_{r=1}^{i} a_r + (i-1) - i = \frac{n+1}{2} - 1 = \frac{n-1}{2}$

and $\begin{bmatrix} n \\ a_1\ a_2 \cdots a_{i-1}\ a_i - 1 \end{bmatrix}_q$ is q-positive

$\therefore$ By Prop. B.3 (c.f. Chromatic quasisymmetric functions), $A_n(q,t)$ is q-unimodal and palindromic.

Order the alphabet $[n] \cup [\bar{n}]$:

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n$$

For $\sigma \in S_n$, define $\bar{\sigma}$ s.t. $\bar{\sigma}(i) = \begin{cases} \overline{\sigma(i)} & \text{if } \sigma(i) > i \quad (i \in EXC(\sigma)) \\ \sigma(i) & \text{if } \sigma(i) \leq i \end{cases}$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix}$

$\therefore \bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{5} & \bar{3} & 1 & 4 & \bar{6} & 2 \end{pmatrix} = (\bar{5}\ \bar{3}\ 1\ 4\ \bar{6}\ 2)$

For $\sigma \in S_n$, define $DEX(\sigma) := DES(\bar{\sigma})$.

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix}$. $\quad \bar{\sigma} = (\bar{5}\ \bar{3}\ 1\ 4\ \bar{6}\ 2)$ $\quad \therefore DEX(\sigma) = DES(\bar{\sigma}) = \{1, 4\}$

Fact (Lemma 2.2 in Eulerian quasisymmetric functions (Shareshian & Wachs)):

$$\sum_{i \in DEX(\sigma)} i = maj(\sigma) - exc(\sigma)$$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix}$. $\quad maj(\sigma) = 1+2+5 = 8$, $\underbrace{exc(\sigma)}_{}$ # bars $= |\{1,2,5\}| = 3$, $\quad \sum_{i \in DEX(\sigma)} i = 1+4 = 5 = 8-3 = maj(\sigma) - exc(\sigma)$.

Define $\cdot\ H(u) := \sum_{n \geq 0} h_n u^n$,

Gessel's Fundamental Quasisymmetric functions (variant)
For $S \subseteq [n-1]$,
$$F_{S,n}(x) := \sum_{\substack{i_1 \geq \cdots \geq i_n \geq 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$
⟵ usual version: $i_1 \leq i_2 \leq \cdots \leq i_n$, $j \in S \Rightarrow i_j < i_{j+1}$

$\left.\begin{array}{l} F_{\phi,n} = h_n \\ F_{[n-1],n} = e_n \end{array}\right\}$

- $Q_n(x,t) = \sum_{\sigma \in S_n} F_{DEX(\sigma)}\, t^{exc(\sigma)}$ (Eulerian quasi-symmetric function)

e.g. $n=3$

$\sigma \in S_3$	$\bar{\sigma}$	$DEX(\sigma)$	$exc(\sigma)$
123	123	ϕ	0
132	$1\bar{3}2$	$\{1\}$	1
213	$\bar{2}13$	ϕ	1
231	$\bar{2}\bar{3}1$	ϕ	2
312	$\bar{3}12$	ϕ	1
321	$\bar{3}21$	$\{2\}$	1

$F_\phi = h_3$

$F_{\{1\}} = \sum_{i<j} x_i^2 x_j + \sum_{i<j<k} x_i x_j x_k$

$F_{\{2\}} = \sum_{i<j} x_i x_j^2 + \sum_{i<j<k} x_i x_j x_k$

$F_{\{1,2\}} = e_3$

$\}\ Q_3(q,t) = (1+2t+t^2) F_\phi + t F_{\{1\}} + t F_{\{2\}}$.

Known identity:

$$\sum_{n\geq 0} Q_n(q,t)\, u^n = \frac{(1-t)\, H(u)}{H(tu) - t\, H(u)}$$ (this implies $Q_n(x,t)$ is symmetric)

Define a variation of the principal specialization of any homogeneous quasisymmetric function f of degree n:

$$ps(f) := f(1,q,q^2,\dots)\cdot \underbrace{(1-q)^n}_{\text{extra piece not in usual } ps(f)}$$

$$* \; ps(F_{S,n}) = \frac{q^{\sum_{s\in S} s}}{[n]_q!}$$

$$\therefore ps(\underbrace{F_{\emptyset,n}}_{h_n}) = \frac{1}{[n]_q!} \qquad \text{e.g. } ps(h_3) = \frac{1}{(1-q)(1-q^2)(1-q^3)}\cdot(1-q)^3 = \frac{1}{(1+q)(1+q+q^2)} = \frac{1}{[3]_q!}$$

$$\therefore ps(\underbrace{F_{[n-1],n}}_{e_n}) = \frac{q^{\binom{n}{2}}}{[n]_q!} \qquad \text{e.g. } ps(e_3) = \frac{q^3}{(1-q)(1-q^2)(1-q^3)}(1-q^3) = \frac{q^3}{[3]_q!}$$

$$\therefore ps(H(u)) = \sum_{n\geq 0} ps(h_n)\, u^n = \sum_{n\geq 0} \frac{u^n}{[n]_q!} = \exp_q(u)$$

Hence $$ps\left(\frac{(1-t)H(u)}{H(tu)-tH(u)}\right) = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$$

$\therefore$ To prove $\displaystyle\sum_{n\geq 0} A_n(q,t)\cdot\frac{u^n}{[n]_q!} = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$, it suffices to show that $\displaystyle ps(Q_n(q,t)) = \frac{A_n(q,t)}{[n]_q!}$.

$$\begin{aligned} ps(Q_n(q,t)) &= ps\Big(\sum_{\sigma\in S_n} F_{\mathrm{DEX}(\sigma)}\, t^{\mathrm{exc}(\sigma)}\Big) \\ &= \sum_{\sigma\in S_n} ps(F_{\mathrm{DEX}(\sigma)})\, t^{\mathrm{exc}(\sigma)} \\ &= \sum_{\sigma\in S_n} \frac{q^{\sum_{i\in \mathrm{DEX}(\sigma)} i}}{[n]_q!}\, t^{\mathrm{exc}(\sigma)} \\ &= \sum_{\sigma\in S_n} \frac{q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}\, t^{\mathrm{exc}(\sigma)}}{[n]_q!} = \frac{A_n(q,t)}{[n]_q!}. \end{aligned}$$

To prove

$$\sum_{n\geq 0} Q_n(q,t)\, u^n = \frac{(1-t)\, H(u)}{H(tu) - t\, H(u)} \qquad \text{———— } (*)$$

We need the following characterizations:

Banner characterization:

A banner is a word over alphabets $\{1,\bar{1},2,\bar{2},\dots\}$ where $1 \leq \bar{1} < 2 \leq \bar{2} < \cdots$ s.t.

- each unbarred letter is followed by a letter $\geq$ in value or unless it is the last entry
- each barred letter is followed by a letter $\leq$ in value

e.g. $b = (2,3,\bar{8},3,3,\bar{5},\bar{5},\bar{4},4,6)$ is a banner word.

Notation: $x_b = \prod_{i\in b} x_i$.
(ignore bars)

eg. $b = (2,3,\bar{8},3,3,\bar{5},\bar{5},\bar{4},4,6)$

$x_b = x_2 x_3^3 x_4^2 x_5^2 x_6 x_8$

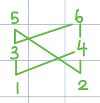
Theorem: $$Q_n(x,t) = \sum_{\substack{b\in B_n \\ \uparrow \\ \text{set of banner words of length } n}} x_b \cdot t^{\#\text{bars in } b}$$ (This gives a recurrence relation for $Q_n(x,t)$ which leads to a generating function formula which proves $(*)$)
(c.f. Thm 1.2 in Eulerian Quasisymmetric functions)

P- partitions :

Given a finite poset P, a function $f : P \to \mathbb{Z}_{>0}$ is a P- partition if it is weakly decreasing,

i.e. $f(a) \geq f(b)$ if $a <_P b$.

e.g. P =

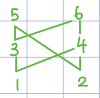


$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 11 & 10 & 9 & 1 & 2 \end{pmatrix}$ is a P- partition.

Given a finite poset P, a function $f : P \to \mathbb{Z}_{>0}$ is a strict P- partition if it is strictly decreasing,

i.e. $f(a) > f(b)$ if $a <_P b$.

e.g. P =



$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 11 & 10 & 9 & 1 & 2 \end{pmatrix}$ is a P- partition, but not strict.

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 10 & 9 & 9 & 1 & 2 \end{pmatrix}$ is a strict P- partition.

Given a poset P, let
- $\mathcal{F}_P$ be the set of all P- partitions.
- $\tilde{\mathcal{F}}_P$ be the set of all strict P- partitions.

Define $K_P(x) := \sum_{f \in \mathcal{F}_P} x_f$, $\tilde{K}_P(x) := \sum_{f \in \tilde{\mathcal{F}}_P} x_f$

Then $K_P, \tilde{K}_P \in QSym_n$ where $n = |P|$.

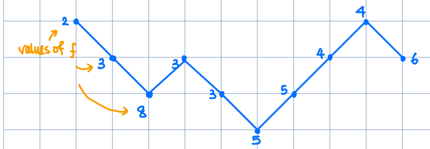
Theorem: (Stanley P-partition reciprocity)

For any poset P of size n,

$\omega K_P(x) = \tilde{K}_P(x)$, where ω is an involution on $QSym_n$ s.t. $\omega : F_{S,n} \leftrightarrow F_{[n-1]\setminus S, n}$

Stanley's observation:

Let P be a zig-zag poset (fence). Given a P- partition $f \in K_P$,



$\in K_P$ (b/c f- value decreases when going up)

We can associate a banner word s.t. number is barred when it goes up :

$2\,3\,\bar{8}\,3\,3\,\bar{5}\,\bar{5}\,\bar{4}\,4\,6 \in B_{10}$

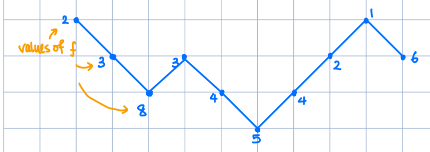
∴ We have :

$$\sum_{b \in B_n} x_n t^{\#bars(b)} = \sum_{P \in \underbrace{ZZ_n}_{\text{zigzag path with n vertices}}} K_P(x)\, t^{\text{upsteps}}$$

Def: A Smirnov word is a word with adjacent letters different

e.g. 2383454216 is a Smirnov word.

Let P be a zig-zag poset (fence). Given a strict P-partition $f \in K_P$,



$\in \widetilde{K}_P$ (b/c f-value strictly decreases when going up)

We can associate a Smirnov word whose descents correspond to up-steps:

2 3 8 3 4 5 4 2 1 6

Let SW_n be the set of all Smirnov words on $\mathbb{Z}_{>0}$ with length n.

Then

$$\sum_{w \in SW_n} x_w t^{\operatorname{des}(w)} = \sum_{P \in ZZ_n} \widetilde{K}_P(x)\, t^{\text{upsteps}}$$

By P-partition reciprocity,

$$\omega\Big(\sum_{P \in ZZ_n} K_P(x)\, t^{\text{upsteps}}\Big) = \sum_{P \in ZZ_n} \widetilde{K}_P(x)\, t^{\text{upsteps}}$$

$$\therefore \quad \omega\Big(\sum_{b \in B_n} x_b t^{\#\text{bars}(b)}\Big) = \boxed{\sum_{w \in SW_n} x_w t^{\operatorname{des}(w)}}$$ ← Define this as $SW_n(x,t)$

i.e. $\omega Q_n(x,t) = SW_n(x,t)$

Since $\displaystyle\sum_{n\geq 0} Q_n(x,t)\, u^n = \frac{(1-t)H(u)}{H(tu)-tH(u)}$, we have (by applying ω on both sides)

$$\sum_{n \geq 0} SW_n(x,t)\, u^n = \frac{(1-t)E(u)}{E(tu)-tE(u)} \quad \text{where } E(u) = \sum_{n\geq 0} e_n u^n.$$

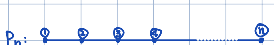
* Smirnov words are just proper colorings of path graphs with n nodes.

e.g. 2 3 8 3 4 5 4 2 1 6 $\in SW_{10}$

← a proper coloring of P

$\therefore$ $SW_n(x,1)$ = chromatic symmetric functions of paths with n nodes.

If we label the nodes of a path with n nodes as:



← label of nodes

$$SW_n(x,t) = \sum_{c \in C(P_n)} t^{\operatorname{des}(c)} x_c$$

proper coloring (John used K_{P_n} in his lectures)