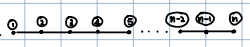
Recall from last time:

$$A_n(q,t) \xrightarrow{\text{lift}} Q_n(x,t) \xrightarrow{\omega} SW_n(x,t)$$

$$SW_n(x,t) := \sum_{w \in S_n} x_w t^{des(w)} = \sum_{c \in C(P_n)} x_c t^{des(c)} = \sum_{c \in C(P_n)} x_c t^{asc(c)}$$

where $P_n =$  is a labeled path with n nodes

How about other graphs instead of path graphs?

Recall the following we learned in previous lectures:

- $X_G(x,t) = \sum_{c \in C(G)} t^{asc(G)} x_c \in QSym_n$ over $\mathbb{Q}[t]$, here
  - $asc(c) = |\{\{i,j\} \in E(G) : i<j \text{ and } c(i)<c(j)\}|$
  - $des(c) = |\{\{i,j\} \in E(G) : i<j \text{ and } c(i)>c(j)\}|$

Prop: If $X_G(x,t)$ is symmetric, then $t^{|E(G)|} X_G(x,t^{-1}) = X_G(x,t)$.

Hence $X_G(x,t)$ is palindromic and $asc(c)$ can be replaced by $des(c)$, i.e. $X_G(x,t) = \sum_{c \in C(G)} t^{des(c)} x_c$.

Q: When is $X_G(x,t)$ symmetric?

We only know "partial" answers:

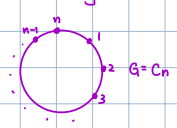
(1) When G is a unit interval graph $inc(P)$, where P is a unit interval order, then $X_G(x,t)$ is symmetric (proved by Shareshian & Wachs)

Choose a finite set of closed intervals $[a_i, a_i+1]$ $(1 \le i \le n)$ of length one on $\mathbb{R}$, with $a_1 < a_2 < \cdots < a_{n-1} < a_n$.

The associated natural unit interval order P is a poset on $\{1,2,\dots,n\}$ in which $i <_P j$ iff $a_i + 1 < a_j$.

e.g.  $\Rightarrow$ $P =$ 

(2) When G is a naturally labeled cycle, then $X_G(x,t)$ is symmetric. (by Ellzey and Wachs)



(3) G has connected components of types (1) or (2), then $X_G(x,t)$ is symmetric.

(We will focus on connected graphs)

Recall $\sum_{n \ge 0} X_{P_n}(x,t) u^n = \dfrac{(t-1)E(u)}{E(tu) - tE(u)}$, we have:

P_n: path with n nodes

- X_{P_n} is e-positive and e-unimodal (i.e. for $0 \le j \le \lfloor \frac{n-1}{2} \rfloor$, coeff of t^{j+1} − coeff of t^j is e-positive)
- Frobenius character of representations of S_n on cohomology of toric variety associated with dual permutation is $\omega X_{P_n}(x,t)$. (by Procesi & Stanley)

Conjecture 1: (Refinement of Stanley-Stembridge)

If G is a natural unit interval graph, then $X_G(x,t)$ is e-positive and e-unimodal.

Conjecture 2: (now proved by Brosnan-Chow and Guay-Paquet)

Connection with Hessenberg varieties

Consequences: Conjecture 1 $\Rightarrow$ Schur-positivity (c.f. proof in Lecture 3 using P-tableaux)

Conjecture 2 (Theorem) + Hard Lefschetz Theorem $\Rightarrow$ Schur-unimodality

Problem: Find a proof of the Schur-unimodality that involves P-tableaux

Conjecture 3: p-positivity and p-unimodality of $\omega X_G(x,t)$ (p: power sum symmetric functions)

Theorem: (Stanley 1995)

For all graphs G, $X_G(x,1) = \sum_{\pi \in \Pi_G} \mu(\hat{0},\pi)\, p_{\mathrm{type}(\pi)}$.

Recall: $\begin{cases} \mu(s,s)=1 & \forall s \in P \\ \mu(s,u) = -\sum_{s \le t < u} \mu(s,t) & \forall s<u \text{ in } P \end{cases}$ P: poset

Π_G: bond lattice of G

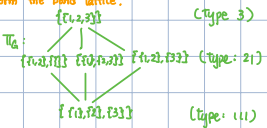
(A bond of G is a partition of its vertices s.t. all vertices in the same part are connected within G.
The set of all bonds of G form the bond lattice.)

Hence $\omega X_G(x,1)$ is p-positive.

eg

G:

Π_G:



(type 3)

(type: 21)

(type: 111)

$$X_G(x,1) = p_{111} - 3p_{21} + 2p_3$$

$$\therefore\ \omega X_G(x,1) = (-1)^{3-3} p_{111} - 3(-1)^{3-2} p_{21} + 2(-1)^{3-1} p_3$$

$$= p_{111} + 3p_{21} + 2p_3 \quad (p\text{-positive})$$

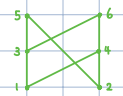
p-positivity of $\omega X_G(x,t)$:

Let P be a poset on $\{1,2,\dots,n\}$.

A word $a_1 a_2 \cdots a_k$ over $\{1,2,\dots,n\}$ has a P-descent at i if $a_i >_P a_{i+1}$, and a left-to-right P-max at i if $a_i >_P a_j$ $\forall j<i$.

Define $\eta_P := \{\sigma \in S_n : \sigma$ has no P-descents or left-to-right P-max (in one-line notation)$\}$.

e.g. $P =$



bigger than both 1,2 in P

left-to-right P-max

$\sigma = 21\boxed{5}436 \notin \eta_P$

P-descent

$\sigma = 43\boxed{5}216 \notin \eta_P$

$\sigma = 432156 \in \eta_P$

Theorem (Shareshian and Wachs)

Let $G = \mathrm{inc}(P)$ where P is a natural unit interval order.

Then coefficient of $\frac{p_n}{n}$ in the power-sum expansion of $\omega X_{\mathrm{inc}(P)}(x,t)$ is $\sum_{\sigma \in \eta_P} t^{\mathrm{inv}_G(\sigma)}$ where $\mathrm{inv}_G(\sigma) = |\{\{\sigma(i),\sigma(j)\} \in E(G) : i<j \text{ and } \sigma(i) > \sigma(j)\}|$.

What about coeff. of $\frac{p_\lambda}{z_\lambda}$?

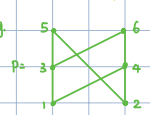
Given a poset P on $\{1,2,\dots,n\}$ and $\lambda \vdash n$.

Define $\eta_{P,\lambda}$ be the set of fillings of Young diagram λ with $1,2,\dots,n$, used once each (i.e. standard filling), s.t. rows have no P-descent and no left-to-right P-max.

If $\lambda = (n)$, then $\eta_{P,\lambda} = \eta_P$ (only one row)

e.g.

$P =$



$\lambda = (3,2,1) \vdash 6$.

$\begin{matrix} 4 & 3 & 2 \\ 5 & 6 & \\ 1 & & \end{matrix} \in \eta_{P,\lambda}$

Theorem (Conjectured by Shareshian & Wachs, proved by Athanasiadis):

Let $G = \mathrm{inc}(P)$ where P is a natural unit interval order, and $\lambda \vdash n$.

Then coefficient of $\frac{p_\lambda}{z_\lambda}$ in the power-sum expansion of $\omega X_G(x,t)$ is $\sum_{T \in \eta_{P,\lambda}} t^{\mathrm{inv}_G(\mathrm{rw}(T))}$ where $\mathrm{rw}(T) \in S_n$ is obtained by reading T from left to right, starting from the top row.

e.g. $\mathrm{rw}\begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & \\ 1 & & \end{pmatrix} = 432561$

e.g. $P =$

$G =$

$(n=4)$ Choose $\lambda = (2,2) \vdash 4$

$$\eta_{P,\lambda} = \left\{ \begin{matrix} 12 \\ 34 \end{matrix}, \begin{matrix} 12 \\ 43 \end{matrix}, \begin{matrix} 21 \\ 34 \end{matrix}, \begin{matrix} 21 \\ 43 \end{matrix}, \begin{matrix} 34 \\ 12 \end{matrix}, \begin{matrix} 34 \\ 21 \end{matrix}, \begin{matrix} 43 \\ 12 \end{matrix}, \begin{matrix} 43 \\ 21 \end{matrix} \right\}$$

$\mathrm{rw}(T)$:	1234	1243	2134	2143	3412	3421	4312	4321
$\mathrm{inv}_G(\mathrm{rw}(T))$:	0	1	1	2	1	2	2	3

$$\therefore \sum_{T \in \eta_{P,(2,2)}} t^{\mathrm{inv}_G(\mathrm{rw}(T))} = 1 + 3t + 3t^2 + t^3.$$

* Note: This is palindromic and unimodal.

* p-unimodality of $\omega X_G(x,t)$ (Conjecture 3) $\Rightarrow \sum_{T \in \eta_{P,\lambda}} t^{\mathrm{inv}_G(\mathrm{rw}(T))}$ is unimodal (This is still open)