Theorem: Let P be a natural unit interval poset and $G = \mathrm{Inc}(P)$.

For $\lambda \vdash n$ (n = #vertices in P), the coefficient c_λ^P of $\frac{p_\lambda}{z_\lambda}$ in the power sum expansion of $\omega X_G(x,t)$ is $\sum_{T \in \eta_{P,\lambda}} t^{\mathrm{inv}_G(rw(T))}$ (denote as $\eta_{P,\lambda}(t)$)

(Athanasiadis' proof):

Ingredients: (1) Observation: relating $\eta_{P,\lambda}(t)$ to $\eta_P(t)$ ← the case when $\lambda = (n)$

(2) Use the result: Coeff. of $\frac{p_n}{n}$ in the power-sum expansion of $\omega X_{\mathrm{Inc}(P)}(x,t)$ is $\sum_{\sigma \in \eta_P} t^{\mathrm{inv}_G(\sigma)}$ (see lecture 6)

(3) Use the decomposition in basis of fundamental quasisymmetric functions.

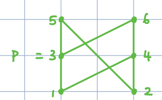
(4) Apply a lemma implicit in work of Roichman which transfers decomposition in fundamental to decomposition in power-sum symmetric functions.

(1) Observation: relating $\eta_{P,\lambda}(t)$ to $\eta_P(t)$

Given $T \in \eta_{P,\lambda}$, let $B(T)$ be the set composition formed by rows of T.

For any graph G on $[n]$ where $n = |P|$, $\mathrm{inv}_G(rw(T)) = \sum_{i=1}^{k} \mathrm{inv}_G(\mathrm{row}_i(T)) + \mathrm{inv}_G(B(T))$ ← k = # rows in T

e.g.



Choose $\lambda = (3,2,1) \vdash 6$

Then $T = \begin{matrix} 4 & 3 & 2 \\ 5 & 6 & \\ 1 & & \end{matrix} \in \eta_{P,\lambda}$ $\therefore$ $B(T) = \{\{2,3,4\}, \{5,6\}, \{1\}\}$

Let G be any graph on $\{1,2,\dots,6\}$ (G does not need to be $\mathrm{Inc}(P)$)

Then $\mathrm{inv}_G(\underbrace{rw(T)}_{432561}) = \mathrm{inv}_G(432) + \mathrm{inv}_G(56) + \mathrm{inv}_G(1) + \underbrace{\mathrm{inv}_G(\overbrace{\{2,3,4\}, \{5,6\}, \{1\}}^{B(T)})}_{}$

i.e. compare and see which of
(2,5), (2,6), (2,1)
(3,5), (3,6), (3,1)
(4,5), (4,6), (4,1)
(5,1),
(6,1)
are inversions in G.

$\therefore \underbrace{\sum_{T \in \eta_{P,\lambda}} t^{\mathrm{inv}_G(rw(T))}}_{\eta_{P,\lambda}(t)} = \sum_{B \in SC_\lambda} t^{\mathrm{inv}_G(B)} \cdot \prod_{i=1}^{\ell(\lambda)} \overbrace{\sum_{\alpha \in \eta_{P_i}} t^{\mathrm{inv}_G(\alpha)}}$

can use result when $\ell(\lambda) = 1$ (see (2))

SC_λ ← set composition of type λ

P_i = poset $P|_{B_i}$

(e.g. In the example above, $P_1 = $ 3 4 2, $P_2 = $ 5 6, $P_3 = $ 1)

(2) $\sum_{\alpha \in \eta_{P_i}} t^{\mathrm{inv}_G(\alpha)}$ = coeff. $c_{(\lambda_i)}^{P_i}$ of $\frac{p_{(\lambda_i)}}{\lambda_i}$ in $X_{\mathrm{Inc}(P_i)}(x,t)$. (by Shareshian & Wachs)

Put (2) into (1), we have (pick $G = \mathrm{Inc}(P)$)

$$\eta_{P,\lambda}(t) = \sum_{B \in SC_\lambda} t^{\mathrm{inv}_G(B)} \prod_{i=1}^{\ell(\lambda)} c_{(\lambda_i)}^{P_i}$$

Hence we want to prove c_λ^P (in the theorem) is $\sum_{B \in SC_\lambda} t^{\mathrm{inv}_G(B)} \prod_{i=1}^{\ell(\lambda)} c_{(\lambda_i)}^{P_i}$.

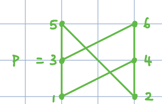
(3) Decomposition into fundamental quasi-symmetric functions:

Theorem: (Shareshian & Wachs; Chow proved the case $t=1$)

Let P be a poset on $[n]$ and $G = \mathrm{Inc}(P)$. Then

$$\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_G(\sigma)} F_{\mathrm{DES}_P(\sigma)}$$ ← Descent set of σ with the partial order $<_P$.

e.g.



$\sigma = (432561)$ $\therefore \mathrm{DES}_P(\sigma) = \{5\}$

(4) Roichman's lemma:

Let $g(x) \in \mathrm{Sym}_n^R$ where R is a commutative $\mathbb{Q}$-algebra. Write $g(x) = \sum_{S \subseteq [n-1]} \alpha_S F_S$ where $\alpha_S \in R$. Then the coeff. of $\frac{p_\lambda}{z_\lambda}$ in the power-sum expansion of $g(x)$ is

$$\sum_{S \in U_\lambda} (-1)^{|S \setminus S(\lambda)|} \alpha_S$$

where $S(\lambda) := \{\lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3, \dots, \lambda_1 + \cdots + \lambda_{\ell(\lambda)-1} = n - \lambda_{\ell(\lambda)}\}$

U_λ = set of λ-unimodal subsets of $[n-1]$.

e.g. $\lambda = (433) \vdash 10$ ($\because n=10$), $S(\lambda) = \{4, 4+3\} = \{4,7\}$

To form a λ-unimodal subset of $[9]$:

$1, 2, 3, 4, 5, 6, 7, 8, 9$

- separate $1, 2, \dots, 9$ into segments by elements in $S(\lambda)$:

$\underbrace{1, 2, 3}, \boxed{4}, \underbrace{5, 6}, \boxed{7}, \underbrace{8, 9}$

3 segments

- In each segment, choose any "initial subsegment":

e.g. $1,2,3$: choices are ϕ, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$

$5,6$: choices are $\phi, \{5\}$, $\{5,6\}$

$8,9$: choices are $\phi, \{8\}, \{8,9\}$

- Choose any subset of $S(\lambda)$: $\{4,7\}$: choices are ϕ, $\{4\}$, $\{7\}$, $\{4,7\}$

Then the union of the chosen initial segments and the chosen subset of $S(\lambda)$ forms a λ-unimodal subset

e.g. $\phi \sqcup \{5,6\} \sqcup \{8\} \sqcup \{4\} = \{4,5,6,8\}$ is $(4,3,3)$-unimodal.

$\{1,2\} \sqcup \{5,6\} \sqcup \phi \sqcup \{7\} = \{1,2,5,6,7\}$ is $(4,3,3)$-unimodal.

$\{2\} \sqcup \{5,6\} \sqcup \phi \sqcup \{4\} = \{2,4,5,6\}$ is not $(4,3,3)$-unimodal

$$\because \ \omega X_{inc(P)}(x,t) = \sum_{\sigma \in S_n} F_{DES_P(\sigma)} t^{inv_G(\sigma)} = \sum_{S \subseteq [n-1]} \underbrace{\Big(\sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) = S}} t^{inv_G(\sigma)} \Big)}_{\alpha_S} F_S$$

$\therefore$ Set $\alpha_S = \sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) = S}} t^{inv_G(\sigma)}$, by Roichman's lemma,

$$c_\lambda^P = \sum_{S \subseteq \mathcal{U}_\lambda} (-1)^{|S \setminus S(\lambda)|} \Big(\sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) = S}} t^{inv_G(\sigma)} \Big)$$

$$= \sum_{\substack{\sigma \in S_n \\ \underbrace{DES_P(\sigma) \in \mathcal{U}_\lambda}_{\text{What does it mean?}}}} (-1)^{|DES_P(\sigma) \setminus DES_P(\sigma)(\lambda)|} t^{inv_G(\sigma)}$$

e.g. $\lambda = (4,3,3) \vdash 10$, $S(\lambda) = \{4,7\}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \boxed{4} & 5 & 6 & \boxed{7} & 8 & 9 & 10 \\ 8 & 5 & 3 & \boxed{10} & 6 & 4 & \boxed{2} & 1 & 7 & 9 \end{pmatrix}$$

We ignore 10 b/c 10 $\notin DES_P(\sigma)$

$\mathcal{U}_\lambda = \{ I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 : I_1 \in \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}, I_2 \in \{\phi, \{5\}, \{5,6\}\}, I_3 \in \{\phi, \{8\}, \{8,9\}\} \}$

e.g. What does it mean if $DES_P(\sigma) = \{1,2,3,4,5,6,7,8,9\}$?

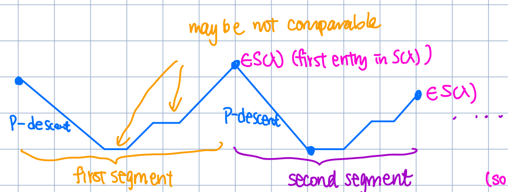
i.e. $8 >_P 5 >_P 3 >_P 10 >_P 6 >_P 4 >_P 2 >_P 1 >_P 7 >_P 9$

What does it mean if $DES_P(\sigma) = \{1,4,5,6,8\}$?

i.e. $8 >_P 5$, $10 >_P 6 >_P 4$, $1 >_P 9$

Hence we are just looking at σ:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \boxed{4} & 5 & 6 & \boxed{7} & 8 & 9 & 10 \\ 8 & 5 & 3 & \boxed{10} & 6 & 4 & \boxed{2} & 1 & 7 & 9 \end{pmatrix}$$



$\leftarrow$ Corresponding P that makes $DES_P(\sigma) \in \mathcal{U}_\lambda$

(so each segment in σ is P-unimodal)

$DES_P(\sigma) \in \mathcal{U}_P$ means
- $8\_5\_3\_10$ We can place $>_P$ from left to right and stop somewhere.
- 10_6
- 6_4_2
- 2_1
- 1_7_9

e.g. $8 \_ 5 \_ 3 \_ 10$ may be incomparable / $<_P$ ($\because$ Not P-descent)

$8 >_P 5 \_ 3 \_ 10$

$8 >_P 5 >_P 3 \_ 10$

$8 >_P 5 >_P 3 >_P 10$

$\therefore$ If $8 <_P 5$ but $3 >_P 10$, then $DES_P(\sigma) \notin \mathcal{U}_\lambda$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \boxed{4} & 5 & 6 & \boxed{7} & 8 & 9 & 10 \\ 8 & 5 & 3 & \boxed{10} & 6 & 4 & \boxed{2} & 1 & 7 & 9 \end{pmatrix}$$

Given $T = \begin{matrix} 8 & 5 & 3 & 10 \\ 6 & 4 & 2 & \\ 1 & 7 & 9 & \end{matrix}$ $\quad (\therefore rw(T) = \sigma)$ s.t. $DES_P(\sigma) \in \mathcal{U}_\lambda$

If we count all P-descents in each row, then we never count the descents in $S(\lambda)$

$T = \begin{matrix} 8 & 5 & 3 & 10 \\ 6 & 4 & 2 & \\ 1 & 7 & \boxed{9} & \end{matrix}$ ← We never consider these entries

always ignore last entry

$\therefore \sum_{i=1}^{\ell(\lambda)} DES_P(row_i(T)) = |S \setminus S(\lambda)|$ where $S = DES_P(rw(T))$.

Let $\mathcal{M}_\lambda^P$ = set of standard fillings of Young diagram of shape λ with P-unimodal rows

Then for $T \in \mathcal{M}_\lambda^P$, $\sum_{i=1}^{\ell(\lambda)} DES_P(row_i(T)) = |DES_P(rw(T)) \setminus DES_P(rw(T))(\lambda)|$

Hence $c_\lambda^P = \sum_{\sigma \in S_n} (-1)^{|DES_P(\sigma) \setminus DES_P(\sigma)(\lambda)|} t^{inv_G(\sigma)}$

$$= \sum_{T \in \mathcal{M}_\lambda^P} (-1)^{\sum_{i=1}^{\ell(\lambda)} DES_P(row_i(T))} t^{inv_G(rw(T))}$$

$$= \sum_{T \in \mathcal{M}_\lambda^P} \left(\prod_{i=1}^{\ell(\lambda)} (-1)^{DES_{P_i}(row_i(T))} \right) t^{\left(\sum_{i=1}^{\ell(\lambda)} inv_G(row_i(T))\right) + inv_G(B)}$$ where B is the block partition formed by rows of T

$$= \sum_{B \in S\mathcal{C}_\lambda} t^{inv_G(B)} \prod_{i=1}^{\ell(\lambda)} \Bigg(\underbrace{\sum_{\substack{\sigma \in S_{B_i} \text{ s.t.} \\ \sigma \text{ is } P_i\text{-unimodal}}} (-1)^{DES_{P_i}(\sigma)} t^{inv_G(B_i)}}_{c_{(\lambda_i)}^{P_i}} \Bigg)$$

← permuting entries in B_i to form $row_i(T)$

For each T, form B by looking at rows of T
For each B, we can form a T by using each block as a row (and choose any permutation of each block)
$\therefore$ For each B, there may be multiple choices of T that can be formed.

$$= \sum_{B \in S\mathcal{C}_\lambda} t^{inv_G(B)} \prod_{i=1}^{\ell(\lambda)} c_{(\lambda_i)}^{P_i}$$ which completes the proof.

Consequences : • Characters of S_n-representation on cohomology of type A semi-simple Hessenberg variety are positive.
• This gives a decomposition of LLT polynomial in p-basis (Carlson - Mellit)
• Other combinatorial description of c_λ^P - Clearman, Hyatt, Shelton, Skandera
- Ellzey - Wachs (generalization to directed graphs)

Theorem : (Shareshian & Wachs, Chow proved the case when $t=1$)

Let P be a poset on $[n]$ and $G = Inc(P)$. Then

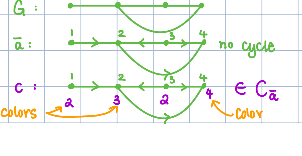
$$\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{inv_G(\sigma)} F_{DES_P(\sigma)}.$$

Proof : (Sketch) Let $AO(G)$ be the set of acyclic orientations of G.

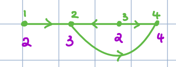
For each $\bar{a} \in AO(G)$, let $C_{\bar{a}}$ be the set of all proper colorings c of G that are compatible with $\bar{a}$,

i.e. $c(i) < c(j)$ if $\{i,j\} \in E(G)$ with $i \to j$ $\quad$ ($\therefore$ coloring goes up if there's an edge)

e.g.



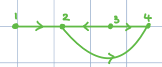
This gives $X_G(x,t) = \sum_{\bar{a}\in AO(G)} t^{asc(\bar{a})} \cdot \sum_{c\in C_{\bar{a}}} x_c$ where $asc(\bar{a}) = |\{(i,j) : i<j \text{ and } i \to j\}|$

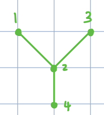
e.g. $\bar{a}$: 

$asc(\bar{a}) = |\{(1,2),(3,4),(2,4)\}| = 3$
$x_c = x_2^2 x_3 x_4$

$\therefore$ This corresponds to the term $t^3 x_2^2 x_3 x_4$ in the sum of $X_G(x,t)$

Note that each $\bar{a} \in AO(G)$ corresponds to a poset $P_{\bar{a}}$.

e.g. $\bar{a}$: 

$P_{\bar{a}}$ =  (Hasse diagram)

(see lecture 5 for definition)

Also, each proper coloring compatible with $\bar{a}$ is the same as a strict $P_{\bar{a}}$-partition.

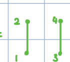
$\therefore \tilde{K}_{P_{\bar{a}}}(x) = \sum_{c\in C_{\bar{a}}} x_c$.

By Stanley's theory of P-partition (not the same as the reciprocity result) which states that:

For any poset P,

$\tilde{K}_P(x) = \sum_{\sigma\in\underbrace{\mathcal{L}(P)}} F_{DES(u\sigma)}$ where u is any fixed decreasing labeling of P

linear extensions of P
(i.e. permutations of $p_1,\dots,p_r \in P$ s.t. $p_i <_P p_j \Rightarrow i<j$)

e.g. P =  $\therefore$ u can be 21 or 43

Then $1234 \in \mathcal{L}(P)$ but $2314 \notin \mathcal{L}(P)$ b/c 2 is before 1 but $1 <_P 2$

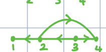
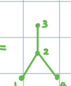
$\therefore \mathcal{L}(P) = \{1234, 1324, 1342, 3124, 3142, 3412\}$

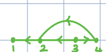
Hence, we have $X_G(x,t) = \sum_{\bar{a}\in AO(G)} t^{asc(\bar{a})} \sum_{\sigma\in\mathcal{L}(P_{\bar{a}})} F_{DES(u_{\bar{a}}P_{\bar{a}})}$

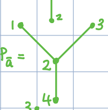
$$= \sum_{\sigma\in S_n} t^{asc(\bar{a}_\sigma)} F_{DES(\underbrace{u_{\bar{a}_\sigma}}\sigma)}$$

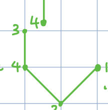
$\bar{a}_\sigma$ is the unique acyclic orientation $\bar{a}$ for which $\sigma\in\mathcal{L}(P_{\bar{a}})$

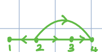
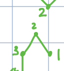
e.g. G = 

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{1423, 4123\}$

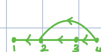
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{2134\}$

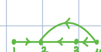
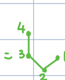
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{4213, 4231\}$

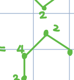
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{2143, 2413, 2431\}$

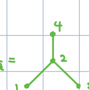
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{1432, 4132, 4312\}$

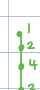
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{4321\}$

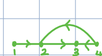
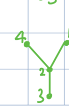
If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{1234\}$

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{2134, 2314, 2341\}$

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{3412, 3142, 1342\}$

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{1324, 3124\}$

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{3421\}$

If $\bar{a}$ = , then $P_{\bar{a}}$ =  $\therefore \mathcal{L}(P_{\bar{a}}) = \{3241, 3214\}$

Each $\sigma\in S_4$ belong to a unique $\mathcal{L}(P_{\bar{a}})$

Hence for $\sigma\in S_4$, there exists exactly one $\bar{a}$ s.t. $\sigma\in\mathcal{L}(P_{\bar{a}})$.

We want to prove $\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{Inv}_G(\sigma)} F_{\mathrm{DES}_P(\sigma)}$.

Since $\omega F_S = F_{[n-1] \setminus S}$, we can rewrite the equation as:

$$X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{Inv}_G(\sigma)} F_{[n-1] \setminus \mathrm{DES}_P(\sigma)}.$$

To finish the proof, we need to verify: (1) $\mathrm{asc}(\bar{a}_\sigma) = \mathrm{Inv}_G(\sigma)$ (This is easy to see. c.f. example above to get an idea.)

(2) there exists a decreasing labeling u of $P_{\bar{a}_\sigma}$ s.t. $\mathrm{DES}(u\sigma) = [n-1] \setminus \mathrm{DES}_P(\sigma)$ (This is hard)