Stanley's Chromatic Symmetric Function (1995)

$G = (V, E)$ (finite, loopless, simple) graph

$K = K_G := \{ \text{Proper colorings } \kappa : V \to \mathbb{Z}_{>0} \}$

$$X_G(\underline{x}) = \sum_{\kappa \in K} x^{\kappa} \in \Lambda$$

$x_1, x_2, \ldots$

$$x^{\kappa} = \prod_{v \in V} x_{\kappa(v)}$$

κ 5 7 5

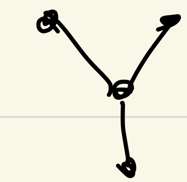
G 

$$x^{\kappa} = x_5^2 x_7$$

$\downarrow$

$$X_G(\underline{x}) = 6\, m_{111} + m_{21}$$
$$= 4\, s_{111} + s_{21}$$
$$= 3\, e_3 + e_{21}$$

s-positive and e-positive

$G =$ 

$$X_G(\underline{x}) = 24m_{1111} + 6m_{211} + m_{31}$$

$$= s_{31} - s_{22} + 5s_{211} + 8s_{1111}$$

Not s or e-positive

Idea: Expand $X_G(\underline{x})$ in terms of familiar bases for Λ, what does this have to do with G?

Sample theorem of Stanley

Write $X_G(\underline{x}) = \sum_{\lambda \in Par(n)} \alpha_\lambda e_\lambda$

$\forall m$
$$\sum_{\ell(\lambda)=m} \alpha_\lambda = \text{\# acyclic orientations of } G \text{ w/ exactly } m \text{ sinks}$$

$X_G(\underline{x})$ contains more information than the chromatic polynomial $\chi_G(t)$

$$\chi_G(n) = X_G(\underbrace{1,1,\dots,1}_{n}, 0, 0, \dots)$$

Chromatic Quasisymmetric Function

Say $V = [n] := \{1, \dots, n\}$

For $\kappa \in K$

$$\mathrm{asc}(\kappa) = \#\{ ij \in E \mid i < j \text{ and } \kappa(i) < \kappa(j) \}$$

$$X_G(\underline{x}; t) := \sum_{\kappa \in K} t^{\mathrm{asc}(\kappa)} x^{\kappa}$$

κ 5 7 5 $\quad \mathrm{asc}(\kappa) = 1 \quad t x_5^2 x_7$

G 1—2—3

$$X_G(\underline{x}; t) = (1 + 4t + t^2) m_{111} + t m_{21}$$

$$= s_{111} + t(2s_{111} + s_{21}) + t^2 s_{111}$$

$$= e_3 + t(e_3 + e_{21}) + t^2 e_3$$

$G = 2 - 1 - 3$

$$X_G(\underline{x};t) = 2(1+t+t^2)\, m_{111} + \sum_{i<j} x_i^2 x_j + t^2 \sum_{i<j} x_i x_j^2$$

$$\in \mathcal{Q}_{\mathbb{Q}[t]} = \mathbb{Q}[t]$$

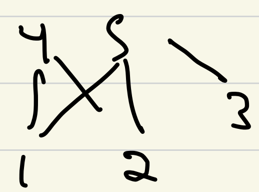
Natural Unit Interval ~~Order~~ Graph

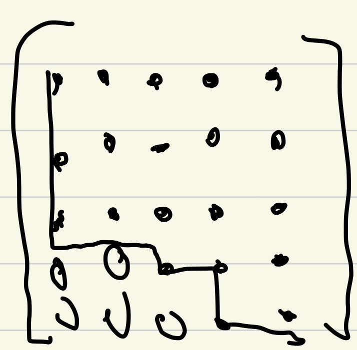
$\underline{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$ is a Hessenberg vector if

- $\underline{m}$ is weakly increasing
- $i \leq m_i \leq n$ for all i.

$\underline{m} = (3,3,4,5,5)$

Poset $P_{\underline{m}}$ on $[n]$, $i <_{P_{\underline{m}}} j$ if $m_i < j$





$G_{\underline{m}} = \mathrm{Inc}(P_{\underline{m}})$ $\qquad\qquad$ $\underline{m} = (2,3,3)$

1 — 2 — 3 — 4 — 5 $\qquad\qquad$ 1 — 2 — 3

<u>Thm</u>: If $\underline{m}$ is a Hessenberg vector, then

$$X_G(\underline{x};t) \in \Lambda[t] = \Lambda_{\mathbb{Q}[t]}$$

Idea: Show that $X_G(\underline{x};t)$ is invariant under each transposition $x_a \leftrightarrow x_{a+1}$

<u>Lemma</u> Given $G = G_{\underline{m}}$ and $\kappa \in K$ and $a > 0$, let

$G_{\kappa,a}$ = Subgraph of G induced on $\kappa^{-1}(a) \cup \kappa^{-1}(a+1)$

Then every connected component $G_{a,\kappa}$ is a path

$$i_1 - i_2 - \cdots - i_j \qquad i_1 < i_2 < \cdots < i_j$$

e.g. $\underline{m} = (3,3,4,5,6,6)$ $\qquad$ 1 — 2 — 3 — 4 — 5 — 6

κ $\quad$ 5 $\;$ 2 $\;$ 6 $\;$ 5 $\;$ 6 $\;$ 2

$G_{\kappa,5} = 1 - 3 - 4 - 5$ $\qquad$ $G_{\kappa,2}$ $\quad$ 2 $\quad$ 6

Proof of lemma • $G_{K,a}$ is bipartite and so contains no 3-cycles.

∴ If $xy, yz \in E$ then $xz \notin E$

Look at $x-y-z$ Goal: either $x<y<z$ or $x>y>z$

Case 1: $x<y$: $y \le m_x$. Either $z \le m_z < x$ or

$x \le m_x < z$

So $x<y\le m_x<z$ Impossible, implies $y > m_z$

Similar argument shows that if $x>y$ then $y>z$.

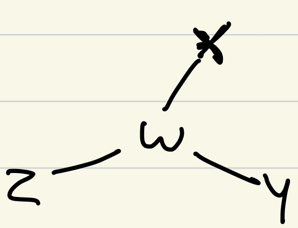
∴ Given a path $i_1 - i_2 - \dots - i_j$ in $G_{K,a}$,

either $i_1<i_2<\dots<i_j$

or $i_1>i_2>\dots>i_j$

∴ $G_{K,a}$ is a forest. Remains to show every vertex

in $G_{K,a}$ has degree < 3



WLOG $x<w<z$ now $x<w<y$

Bad news for $y-w-z$ ∎

Proof of Thm: Involution ~~of~~ on K, given a:

Given connected component $i_1 - i_2 - \cdots i_j$ of $G_{K,a}$

do nothing if j is even, exchange colors

$a, a+1$ if j is odd