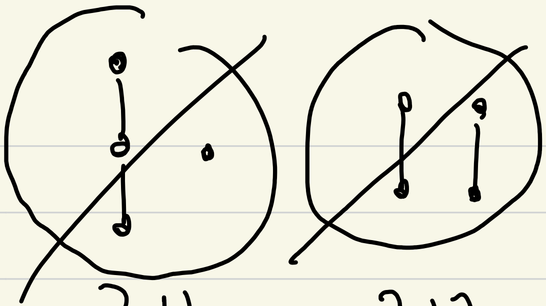


3+1 2+2

$G_{\underline{m}} = \mathrm{Inc}(P_{\underline{m}})$

(Weak) Stanley-Stembridge Conjecture:

$X_{G_{\underline{m}}}(\underline{x})$ is e-positive

(Strong) Stanley-Stembridge Conjecture: If P is (3+1)-free, then $X_{\mathrm{Inc}(P)}(\underline{x})$ is e-positive

Thm (Guay-Paquet): Weak $\Rightarrow$ Strong

Graded Stanley-Stembridge Conjecture (S-Wachs)

$X_{G_{\underline{m}}}(x;t)$ is e-positive

$\underline{m} = (2,3,3)$ $\quad X_{G_{\underline{m}}}(\underline{x};t) = e_3 + t(e_3 + e_{21}) + t^2 e_3$

Frobenius Characteristic

$CF_n = \{$ Class functions $\chi : S_n \to \mathbb{Q} \}$

Cl_λ = Conjugacy class $\{ w \in S_n \mid w$ has cycle shape $\lambda \}$

$\mathbb{1}_{Cl_\lambda}$ = indicator function for Cl_λ

Frob. Char.

$ch : CF_n \to \Lambda^n$

linear $\quad \mathbb{1}_{Cl_\lambda} \mapsto \dfrac{p_\lambda}{z_\lambda}$

(p_λ ← power sum; z_λ ← $|C_{S_n}(w)|$, $w \in Cl_\lambda$)

$ch^{-1}(s_\lambda) = \chi^\lambda$ (Irreducible char.)

$ch^{-1}(h_\lambda)$ = Permutation character on cosets

$S_{\lambda_1} \times \cdots \times S_{\lambda_l}$

$ch^{-1}(e_\lambda) = ch^{-1}(h_\lambda) \otimes \mathrm{sgn}$

· ch is an isometry

$$\langle f, g \rangle = \frac{1}{n!} \sum_{w \in S_n} ch^{-1}(f)(w)\, ch^{-1}(g)(w)$$

$\in \Lambda^n$

Λ

$$\langle s_\lambda, s_\mu \rangle = \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

$$ch(\tau \otimes sgn) = \omega(ch(\tau)) \qquad \omega(e_\lambda) = h_\lambda$$

Immanants

$A = (a_{ij})$

$\tau \in CF_n$

$$\mathrm{Imm}_\tau(A) = \sum_{w \in S_n} \tau(w) \prod_{i=1}^{n} a_{i w_i}$$

$\tau = \chi^{(1,\dots,1)}$ $\qquad \mathrm{Imm}_\tau = \mathrm{Det}$

$\tau = \chi^{(n)}$ $\qquad \mathrm{Imm}_\tau = \mathrm{Per}$

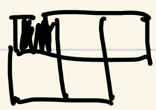
$\tau = \chi^{(2,1)}$ $\qquad \mathrm{Imm}_\tau(A) = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$

Jacobi-Trudi matrices

Partition $\nu \subseteq \mu$, $\ell(\mu) \leq n$ $|\mu/\nu| = N$

$$H(\mu/\nu) = \left(h_{\mu_i - \nu_j + j - i} \right)_{i=j=1}^{n}$$

e.g. $n=3$ $\mu(3,2) = (3,2,0)$ $\nu(1) = (1,0,0)$

μ/ν  $N=4$

$$H(\mu/\nu) = \begin{bmatrix} h_2 & h_4 & h_5 \\ 1 & h_2 & h_3 \\ 0 & 0 & 1 \end{bmatrix}$$

Jacobi-Trudi Identity $s_{\mu/\nu} = \mathrm{Imm}_{\chi^{(1,\dots,1)}} H(\mu/\nu)$

$\mathrm{Deg}\ \mathrm{Imm}_{\tau}(H(\mu/\nu)) = N$

- Conjecture (Goulden–Jackson) / Theorem (Greene):

  $\mathrm{Imm}_{\chi^\lambda}(H(\mu/\nu))$ is m-positive

- Conjecture (Stembridge) / Theorem (Haiman)

  $\mathrm{Imm}_{\chi^\lambda}(H(\mu/\nu))$ is s-positive

- Set $\phi^\lambda = ch^{-1}(m_\lambda)$

<u>Conjecture</u> (Stembridge)

  $\mathrm{Imm}_{\phi^\lambda}(H(\mu/\nu))$ is m-positive

<u>Stronger Conjecture</u> (Stembridge)

  $\mathrm{Imm}_{\phi^\lambda}(H(\mu/\nu))$ is s-positive

$$H = \begin{bmatrix} h_2 & h_4 & h_5 \\ 1 & h_2 & h_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$m_{21} = s_{21} - 2s_{111}$

$\chi^{(2,1)} - \chi^{(1,1,1)}$

$$\begin{array}{rrr} 2 & 0 & -1 \\ -\ 2 & -2 & 2 \\ \hline 0 & 2 & -3 \end{array}$$

Only 123, 213 "count"

$\text{Imm}_{\varphi^{21}}(H) = h_{22} + h_{4}$ Schur positive

$2h_4$

$\delta = (n-1, n-2, \ldots, 2, 1, 0)$

Observe $w \in S_n$

$$\prod_{i=1}^{n} H(\mu,\nu)_{i w_i} = h_{\mu+\delta - w(\nu+\delta)}$$

$(3,2)$ ↖

$\mu+\delta = (5,3,0)$
$\nu+\delta = (3,1,0)$

$w = 123$ $(2,2,0)$
$w = 213$ $(4,0,0)$
rest has a negative entry

$\therefore$

$$\text{Imm}_{\varphi^\lambda}(H(\mu/\nu)) = \sum_{w \in S_n} \varphi^\lambda(w)\, h_{\mu+\delta - w(\nu+\delta)}$$

$$= \sum_{w \in S_n} \varphi^\lambda(w) \sum_{\theta \vdash N} K_{\theta,\, \mu+\delta - w(\nu+\delta)}\, s_\theta$$

← Kostka numbers

Given θ

Set $\Gamma^{\Theta}_{\mu,\nu}(w) = |C_{S_n}(w)| \sum_{x \in Cl(w)} K_{\Theta, \mu+\delta - x(\nu+\delta)} \in CF_n$

Then $\langle \mathrm{Imm}_{\phi^\lambda}(H(\mu/\nu)), S_\Theta \rangle_\Lambda$

$= \langle \phi^\lambda, \Gamma^{\Theta}_{\mu,\nu} \rangle_{S_n} = \langle m_\lambda, ch(\Gamma^{\Theta}_{\mu,\nu}) \rangle_\Lambda$

want this ≥ 0 always $\iff$ $ch(\Gamma^{\Theta}_{\mu,\nu})$ is h-positive

<u>Proposition</u> (Stanley–Stembridge): ← case $\Theta = (N)$

Given μ/ν ~~[illegible]~~ and n, there is some Hessenberg vector $\underline{m}$ such that

$$\Gamma^{(N)}_{\mu,\nu} = \omega X_{G_{\underline{m}}}(x)$$

<u>Conjecture</u> (Stanley–Stembridge) Given $\Theta \vdash N$ and $\nu \subseteq \mu$, there exist $\mu^1/\nu^1, \mu^2/\nu^2, \ldots, \mu^k/\nu^k$ such that

$$\Gamma^{\Theta}_{\mu,\nu} = \sum_{j=1}^{n} \Gamma^{(N)}_{\mu^j, \nu^j}.$$