Thm: Let $\underline{m}$ be a Hessenberg vector. Then

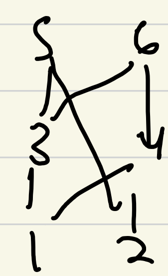
$$X_{G_{\underline{m}}}(\underline{x};t) \text{ is Schur positive.}$$

(Proved Gasharov for (3+1)-free posets when $t=1$)

More specific: Given poset P on $[n]$ and $\lambda \vdash n$, a <u>P-tableau</u> of shape λ is a filling of the Young diagram of shape λ with $1,2,\dots,n$ used once each, so that

- rows are P-increasing
- columns have no P-decrease

e.g. $\underline{m}=(2,3,4,5,6,6)$ $\quad P_{\underline{m}}$



$G_{\underline{m}}$: $1-2-3-4-5-6$

$\lambda=(3,2,1)$

$$\begin{array}{ccc} 2 & 4 & 6 \\ 1 & 3 & \\ 5 & & \end{array}$$

$\mathrm{inv}_{G_{\underline{m}}}(T)=3$

An <u>inversion</u> in a P-tableau T is a pair $\{i<j\}$ with $ij \in E(G_{\underline{m}})$ and i is in a lower row than j in T

$\mathcal{T}_{P,\lambda} = \{ P\text{-tableaux of shape } \lambda \}$

<u>Theorem</u>: $X_{G_{\underline{m}}}(\underline{x};t) = \sum_{\lambda \vdash n} s_\lambda \left(\sum_{T \in \mathcal{T}_{P_{\underline{m}},\lambda}} t^{\mathrm{inv}_G(T)} \right)$

$\underline{m} = (2,3,3)$ $\quad G_{\underline{m}} = 1 - 2 - 3$ $\quad P_{\underline{m}}$ $\begin{array}{cc} 3 & \\ | & 2 \\ 1 & \end{array}$

$\lambda = (3)$ No P_m-tableaux

$\lambda = (2,1)$ $\quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$ $\quad \mathrm{inv}_{G_m}(T) = 1$

$\lambda = (1,1,1)$

	1	1	2	3
	2	3	1	2
	3	2	3	1
Inv	0	1	1	2

$X_{G_{\underline{m}}}(\underline{x};t) = s_{111} + t(2s_{111} + s_{21}) + t^2 s_{111}$

For proof, need to use

instead of $\mathrm{asc}(\kappa)$

$$X_{G_{\underline{m}}}(\underline{x};t) = \sum_{\kappa \in K} \underline{x}^{\kappa}\, t^{\mathrm{des}(\kappa)}$$

After Gasharov:

Given a weak composition $(\alpha_1, \dots, \alpha_n)$ of n, a $P_{\underline{m}}$-array of shape is a partition $\pi_1 | \pi_2 | \cdots | \pi_n$ of $[n]$ so that

- $|\pi_i| = \alpha_i$ and
- π_i is a chain in $P_{\underline{m}}$

"$P_{\underline{m}}$-tableau" of shape α

e.g. $\underline{m} = (2,3,4,5,6,6)$ $\quad \alpha = (1,2,0,2,1)$

$$\begin{array}{l} 6 \\ 1\,3 \\ \\ 2\,5 \\ 4 \end{array}$$

Define inversions as above

Every P-tableau is a P-array

Write $X_{G_{\underline{m}}}(x;t) = \sum_{\lambda \vdash n} \underbrace{c_\lambda(t)}_{\text{What we want to compute}} s_\lambda$

Given $w \in S_n$ define (for $\lambda \in \mathrm{Par}(n)$)

$$w(\lambda) = \left(\lambda_{w(i)} - w(i) + i \right)_{i=1}^{n}$$

~~e.g. $\lambda = (3,3,1)$~~

e.g. $\lambda = (2,1,0)$

w	$w(\lambda)$		
123	$(2,1,0)$		
132	$(2,-1,2)$	X	Doesn't count
213	$(0,3,0)$		Counts
$\vdots$			

~~Claim: $X_\lambda(t) =$~~

$$\mathcal{A}_{P_{\underline{m}},\lambda} = \{ (A, w) \mid w \in S_n,\ A\ P_{\underline{m}}\text{-array of shape } w(\lambda) \}$$

e.g. $\lambda = (1,1,1)$ $\quad \underline{m} = (2,3,3)$ $\quad P_{\underline{m}} = \begin{array}{cc} 3 & \\ | & 2 \\ 1 & \end{array}$

w	$w(\lambda)$
123	$(1,1,1)$
132	$(1,0,2)$
213	$(0,2,1)$
231	$(0,0,3)$
312	$(-1,2,2)$ ✗
321	$(-1,0,3)$ ✗

$$\begin{array}{cccccc} \mathrm{Inv} \; 0 & 1 & 1 & 1 & 1 & 2 \\ \begin{bmatrix}1\\2\\3\end{bmatrix} & \begin{bmatrix}1\\3\\2\end{bmatrix} & \begin{bmatrix}2\\1\\3\end{bmatrix} & \begin{matrix}2\\3\\1\end{matrix} & \begin{matrix}3\\1\\2\end{matrix} & \begin{bmatrix}3\\2\\1\end{bmatrix} \end{array}$$

$$\begin{matrix} 2 \\ \hline 13 \end{matrix} \quad 1 \qquad\qquad \begin{matrix} \hline 13 \\ 2 \end{matrix} \quad 1 \qquad\qquad \cancel{\emptyset}$$

Claim: $$c_\lambda(t) = \sum_{(A,w)\in A_{P,\lambda}} \mathrm{sgn}(w)\, t^{\mathrm{inv}_G(A)}$$

In our example,

$$c_{111}(t) = \underset{123}{1} + 4t + t^2 - \underset{\substack{132\\213}}{2t} = 1+2t+t^2$$

Comment: Each array corresponds to a proper coloring with colors $1,2,\dots,n$ - the elements in row i get color i

How does the claim imply the theorem?

• If $A \in \mathcal{T}_{P_{\underline{m}},\lambda}$ then $(A,1) \in \mathcal{A}_{P_{\underline{m}},\lambda}$

• If $w \neq 1$, then $w(\lambda)$ is not a partition

$\therefore$ If $(A,w) \in \mathcal{A}_{P_{\underline{m}},\lambda}$ then $A \notin \mathcal{T}_{P,\underline{m}}$

Define

$$\mathcal{B}_{P_{\underline{m}},\lambda} := \mathcal{A}_{P_{\underline{m}},\lambda} \setminus \{(A,1) \mid A \in \mathcal{T}_{P,\underline{m}}\}$$

Find an inv_G-preserving, sign-reversing involution on $\mathcal{B}_{P_{\underline{m}},\lambda}$

Given $(A,w) \in \mathcal{B}_{P_{\underline{m}},\lambda}$, call an entry α of A "bad" if either

- There is no entry of A directly above α, or
- The entry β directly above α satisfies $\beta >_{P_{\underline{m}}} \alpha$

Find the leftmost column c in A containing a bad entry, in column c find the lowest bad entry, let r be the row above this lowest bad entry

$c=1$

$r=2$ $\quad$ 2

$\quad$ —

bad 1 3 bad

Let $C_r(A)$ be the set of entries in row r weakly to the right of column c

$C_{r+1}(A)$ be the set of entries in row $r+1$ strictly to the right of column c

$H(A)$ = subgraph of $G_{\underline{m}}$ induced on $C_r(A) \cup C_{r+1}(A)$

(bipartite)

For $i \in \{r, r+1\}$

$O_i(A) := \{\alpha \in C_i(A) \mid \alpha$ is in a component of odd size in $H(A)\}$

$E_i(A) := C_i(A) \setminus O_i(A)$

$I_i(A) := \{$entries in row i but not in $C_i(A)\}$

<u>Claim</u>: We can exchange $O_r(A)$ with $O_{r+1}(A)$ and replace w with $w(r, r+1)$

To prove this, need that $I_r(A) \cup E_r(A) \cup O_{r+1}(A)$ is a chain in $P_{\underline{m}}$

To show: if $\alpha \in I_r(A)$ and $\beta \in O_{r+1}(A)$, then

$$\alpha <_{P_m} \beta$$

$$\begin{array}{ccc} & & \wedge \\ r & \alpha & \nu \\ r+1 & \delta & \bullet\!\!-\!\!-\!\!-\beta\!-\!- \\ & & \varepsilon\uparrow \end{array}$$

"bad"

$$\nu > \varepsilon$$

$$\delta < \varepsilon < \beta$$

Since P_m is $(3+1)$-free, α is related to one of $\delta, \varepsilon, \beta$.

But $\alpha \not>_{P_{\underline{m}}} \delta$

Why does the claim hold?

$$c_\lambda(t) = \left\langle X_G(\underline{x};t),\ s_\lambda \right\rangle$$

$$\overset{\text{Jacobi–Trudi}}{=} \sum_{w \in S_n} \mathrm{sgn}(w) \left\langle X_G(x;t),\ h_{w(\lambda)} \right\rangle$$

$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$

$$= \sum_{w \in S_n} \operatorname{sgn}(w) \left(\text{coeff of } \prod_{i=1}^{n} x^{w(\lambda)_i} \text{ in } X_{G_m}(\underline{x}; t) \right)$$

$$= \sum_{w \in S_n} \operatorname{sgn}(w) \sum_{\kappa \in K_{w(\lambda)}} t^{\operatorname{des}(\kappa)}$$

Proper colorings satisfying $|\kappa^{-1}(i)| = w(\lambda)_i$

This is ~~…~~ what we say in the claim