We saw $X_{G_{\underline{m}}}(x;t)$ is s-positive & therefore the Frob. Char. of a rep. of S_n. What rep?

Flag variety:

$$G = GL_n(\mathbb{C}), \quad B = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \leq G$$

$Fl_n = G/B$ "coset model" projective variety

$Fl_n = \{ V_\bullet = 0 < V_1 < \dots < V_{n-1} < \mathbb{C}^n \mid \dim V_i = i \}$

G acts transitively on the set of all flags, and B is the stabilizer of

$$E_\bullet = 0 < \langle e_1 \rangle < \langle e_1, e_2 \rangle < \dots < \langle e_1, \dots, e_{n-1} \rangle < \mathbb{C}^n$$

"flag model"

Hessenberg Varieties (De Mari-Shayman, De Mari-Procesi-Shayman)

$\underline{m} = (m_1, \dots, m_n)$ Hessenberg vector

$H_{\underline{m}}$ Hessenberg space $\subseteq \mathfrak{gl}_n(\mathbb{C})$

$$H_{\underline{m}} = \{ (a_{ij}) \mid a_{ij} = 0 \text{ if } i > m_j \}$$

$$\underline{m} = (2,4,5,6,6,6)$$

$$H_{\underline{m}} = \begin{bmatrix} * & * & * & & & \\ * & * & * & & & \\ 0 & * & * & & * & \\ 0 & * & * & & & \\ 0 & 0 & * & & & \\ 0 & 0 & 0 & & & \end{bmatrix}$$

$s \in \mathfrak{gl}_n(\mathbb{C})$

$$\mathrm{Hess}(s,\underline{m}) = \begin{cases} \{ gB \mid g^{-1} s g \in H_{\underline{m}} \} \\ \{ V_\bullet \mid s V_j \subseteq V_{m_j} \ \forall j \} \end{cases}$$

$$\begin{aligned} g^{-1} s g \in H_{\underline{m}} &\iff g^{-1} s g (e_j) \in \langle e_1, \dots, e_{m_j} \rangle \\ &\iff s g (e_j) \in g \langle e_1, \dots, e_{m_j} \rangle \\ &\iff s(g(E_\bullet)) \subseteq g(E_\bullet)) \end{aligned}$$

Examples: If $\underline{m} = (n, \ldots, n)$ or $S = \lambda I$

then $\mathrm{Hess}(S, \underline{m}) = Fl_n$

- If $\underline{m} = (1, 2, \ldots, n)$ and S is nilpotent, then $\mathrm{Hess}(S, \underline{m})$ is Springer Fiber
- If $\underline{m} = (2, 3, \ldots, n, n)$ and S is regular nilpotent then $\mathrm{Hess}(S, \underline{m})$ is a Peterson variety
- If $\underline{m} = (2, 3, \ldots, n, n)$ and S is regular semisimple then $\mathrm{Hess}(S, \underline{m})$ is the toric variety associated to the fan of Weyl chambers (De Mari-Procesi-Shayman)

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e.g. $\underline{m} = (2, 3, 3)$ $\quad S = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$ $\quad \{ V_\bullet = V_1 < V_2 < \mathbb{C}^3 \mid S V_1 < V_2 \}$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix}$$

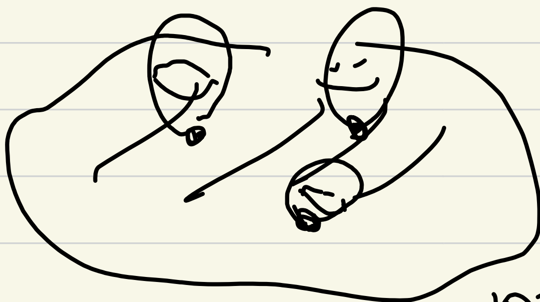
$\pi : \mathrm{Hess}(S, \underline{m}) \to \mathbb{P}^2 \quad 0 < V_1 < V_2 < \mathbb{C}^3 \mapsto V_1$

Given $V_1 \in \mathbb{P}^2$, if V_1 is not an eigenspace for S,

then $\pi^{-1}(V_1) = \{ 0 < V_1 < V_1 + SV_1 < \mathbb{C}^3 \} = \bullet$

If V_1 is an eigenspace (i.e. $V_1 = \langle e_i \rangle$) then

$$\pi^{-1}(V_1) = \{0 < V_1 < X < \mathbb{C}^3 \mid X \supset V_1\} \cong \mathbb{P}^1$$



$\mathbb{P}^2$

$\beta_0 = 1 \quad \beta_2 = 4 \quad \beta_4 = 1$

<u>Thm</u> (De Mari–Shayman, De Mari–Procesi–Shayman, Tymoczko, Precup)

$H^k(\mathrm{Hess}(s, \underline{m})) = 0$ if k is odd and s is regular

$\boxed{\text{Assume for now } s \text{ is regular semisimple}}$

Assume s is diagonal

$$C_G(s) = T = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

T acts on $\mathrm{Hess}(s, \underline{m})$ ← arbitrary

$$sV_j \subseteq V_{m_j} \Rightarrow s(tV_j) = tsV_j \subseteq tV_{m_j}$$

$\therefore$ If $V_\bullet \in \mathrm{Hess}(s,\underline{m})$ then $tV_\bullet \in \mathrm{Hess}(s,\underline{m})$

The action of T on $\mathrm{Hess}(s,\underline{m})$ is equivariantly formal as defined by Goresky-Kottwitz-MacPherson.

Moment Graph M

$$\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

$V(M)$ = vertices $\leftrightarrow$ T-fixed points in Hess $\leftrightarrow$ S_n

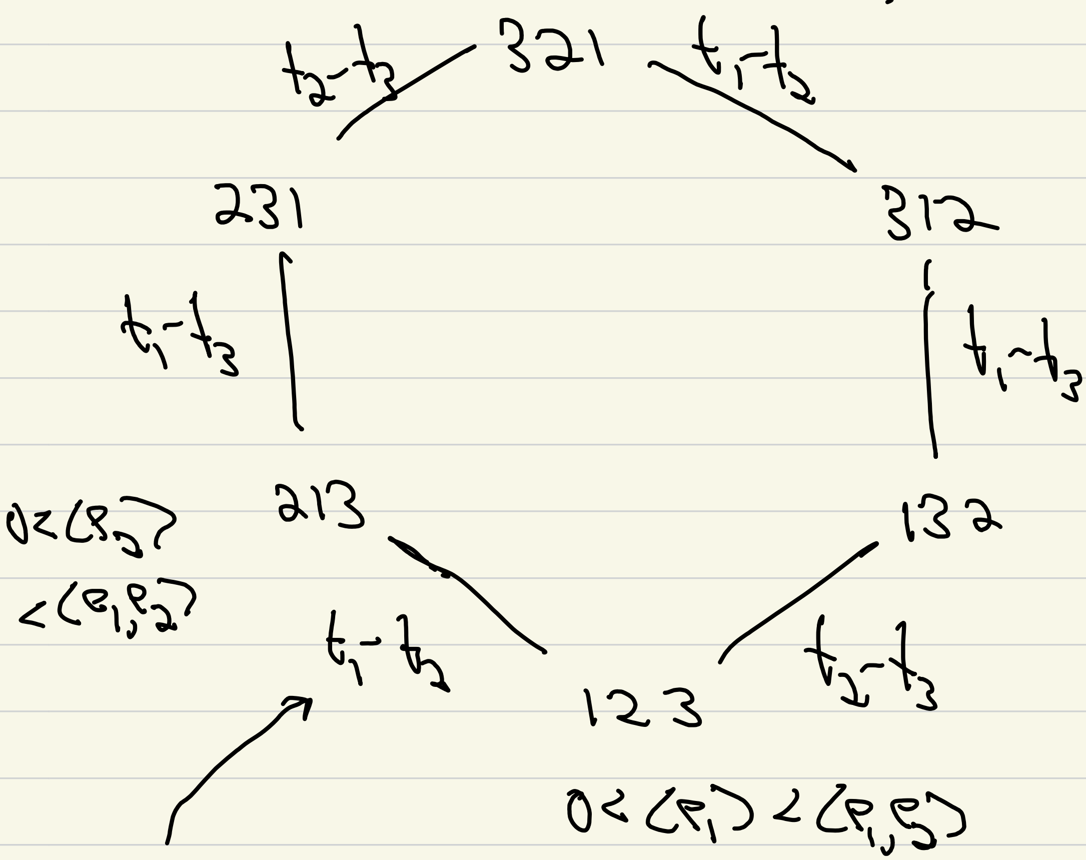
$w \in S_n$ $\quad V_\bullet(w) = 0 \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \cdots$
$\in \mathrm{Hess}(s,\underline{m})$

$E(M)$ = edges $\leftrightarrow$ 1-dimensional T-orbits, bounded by two fixed points

labeled: The stabilizer of a point in a 1-dim'l orbit is a codimension 1 torus, whose Lie algebra is the kernel of some linear form ℓ

$$\bullet \xrightarrow{\ \ell\ } \bullet$$

e.g. $\underline{m} = (2,3,3)$ $\quad S = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$



Orbit of $\langle e_1+e_2\rangle < \langle e_1,e_2\rangle$

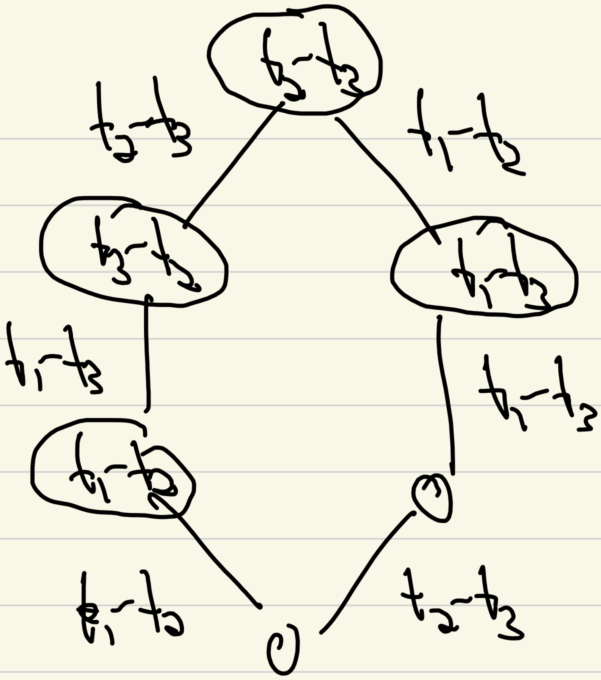
$$\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \langle e_1+e_2\rangle = \{\langle ae_1+be_2 \mid a,b\neq 0\rangle\}$$

stabilizer $\begin{bmatrix} a & & \\ & a & \\ & & b \end{bmatrix}$

GKM Theory says

$$H_T^*(\mathrm{Hess}(S,\underline{m})) \cong \Big\{ f: V(M)\to\mathbb{C}[t_1,\dots,t_n] \;\Big|\; \text{if } a \overset{\ell}{\text{———}} b \in E(M), \text{ then } \ell \text{ divides } f(a)-f(b) \Big\}$$

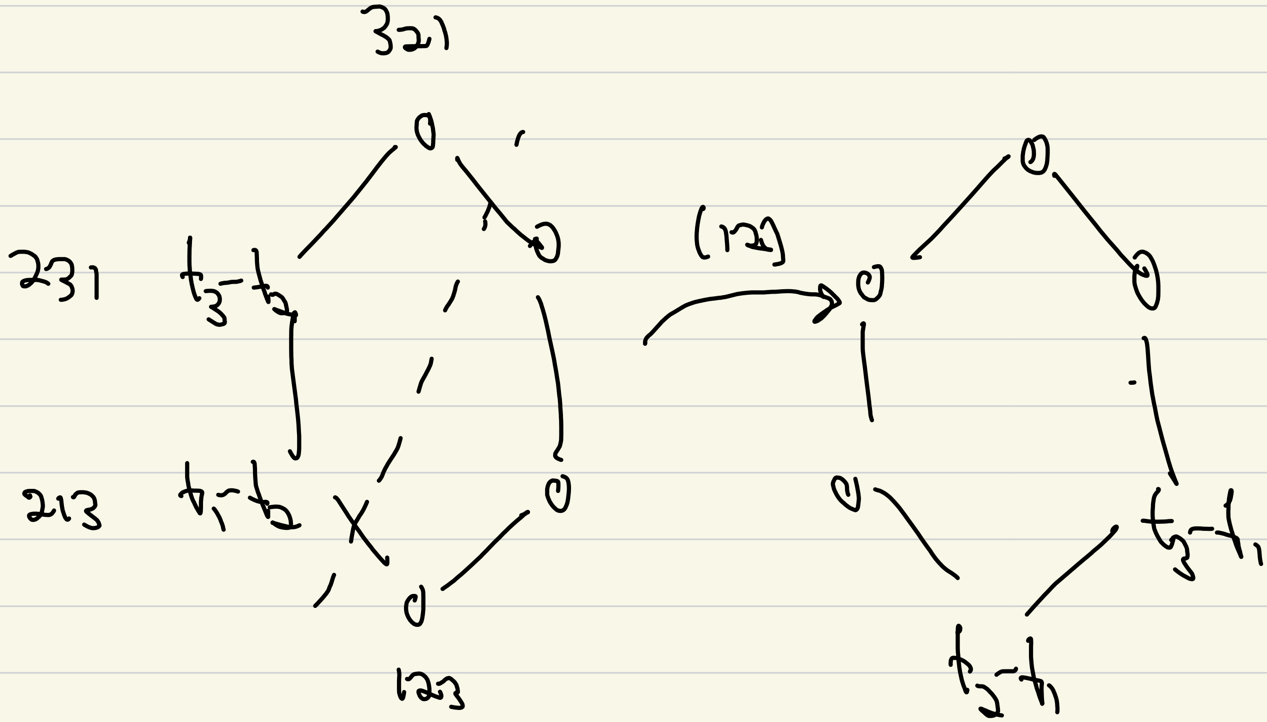
Deg $t_1 = 2$



$\in H^3_T(-)$

H^*_T is a $\mathbb{C}[t_1, \dots, t_n]$,

$H^* \cong H^*_T / (t_1, \dots, t_n) H^*_T$



This gives an action of S_n on H^*_T and H^*

action on indices

and action on itself by multiplication

<u>Conjecture</u> (S-Wachs) / <u>Theorem</u> (Brosnan-Chow) [Guay-Paquet]

The Frobenius characteristic of $H^*(\mathrm{Hess}(S,\underline{m}))$ is

$\omega X_{G_{\underline{m}}}(\underline{x};t)$ Fig. S.S.

$\omega(h_n) = e_n \quad \omega(s_\lambda) = s_{\lambda^t}$

Graded-Stanley Stembridge Conjecture:

The $\mathbb{C}[S_n]$-module $H^{2k}(\mathrm{Hess}(s,\underline{m}))$ has a $\mathbb{C}$-basis permuted by S_n in which the stabilizer of each basis vector is a Young subgroup