Prequel –

Eulerian Polynomials

$$A_n(t) = \sum_{\sigma \in S_n} t^{des(\sigma)} = \sum_{\sigma \in S_n} t^{exc(\sigma)}$$

Recall

Eulerian

- $des(\sigma) = |\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}|$
- $exc(\sigma) = |\{i \in [n-1] \mid \sigma(i) > i\}|$

- $DES(\sigma) = \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$

Mahonian

- $maj(\sigma) = \sum_{i \in DES(\sigma)} i$
- $inv(\sigma) = |\{(i,j) \mid \sigma(i) > \sigma(j),\ i < j\}|$

Ex $A_3(t) = 1 + 4t + t^2$

$A_4(t) = 1 + 11t + 11t^2 + t^3$

Properties of $A_n(t) = \sum_{j=0}^{n-1} \left\langle {n \atop j} \right\rangle t^j$

- palindromic $\left\langle {n \atop j} \right\rangle = \left\langle {n \atop n-1-j} \right\rangle \ \forall j$
- unimodal

$$\left\langle {n \atop 0} \right\rangle \le \left\langle {n \atop 1} \right\rangle \le \dots \le \left\langle {n \atop m} \right\rangle \ge \dots \ge \left\langle {n \atop n-1} \right\rangle$$

q-analog (maj, exc)

$$A_n(q,t) = \sum_{\sigma \in S_n} q^{maj(\sigma) - exc(\sigma)} t^{exc(\sigma)}$$

$$A_3(q,t) = 1 + (2 + q + q^2)t + t^2$$

$$A_4(q,t) = 1 + (3 + 2q + 3q^2 + 2q + q^4)t + (3 + 2q + 3q^2 + 2q + q^4)t^2 + t^3$$

Th (Shareshian & W 2007)

$$\sum_{n \ge 0} A_n(q,t) \frac{u^n}{[n]_q!} = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$$

where $[n]_q = 1 + q + \dots + q^{n-1} = \dfrac{q^n - 1}{q - 1}$

$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$

$$\exp_q(u) = \sum_{n \geq 0} \frac{u^n}{[n]_q!}$$

Note (1) $q=1$ Euler's exp gen function

(2) Follows from formula that

- $A_n(q,t)$ is palindromic
- $A_n(q,t)$ is q-unimodal

Symmetric function identity

$$\sum_{n \geq 0} Q_n(x,t)\, u^n = \frac{(1-t)\, H(u)}{H(tu) - t\, H(u)}$$

$$\Big\downarrow ps \qquad\qquad\qquad \Big\downarrow ps$$

$$\sum_{n \geq 0} \frac{A_n(q,t)}{[n]_q!} u^n = \frac{(1-t)\exp_q(u)}{\exp_q(tu) - t\exp_q(u)}$$

where $H(u) = \sum_{n \geq 0} h_n u^n$ complete homogeneous

$$ps(\overset{Q(x,t)}{f}) = f(1, q, q^2, \dots)\,(1-q)^n$$

Gessel's Fund Quasisymm Function

For $S \subseteq [n-1]$

$$F_{S,n}(x) := \sum_{\substack{i_1 \geq \cdots \geq i_n \geq 1 \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

basis for $QSym_n$

$$ps(F_{S,n}) = \frac{q^{\Sigma S}}{[n]_q!} \qquad \Sigma S := \sum_{s \in S} s$$

Ex $h_n = F_{\emptyset,n}$ $\quad e_n = F_{[n-1],n}$

$$ps(H_n) = ps\left(\sum_{n \geq 0} h_n u^n\right)$$

$$= \sum_{n \geq 0} \frac{u^n}{[n]_q!} = \exp_q(u)$$

<u>Our def</u>

$$Q_n(x,t) = \sum_{\sigma \in S_n} F_{DEX(\sigma)} t^{exc(\sigma)}$$

Eulerian quasisymm function

$$\sum DEX(\sigma) = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$$

$$ps(Q_n(x,t)) = \frac{\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}}{[n]_q!}$$

$$= \frac{A_n(q,t)}{[n]_q!}$$

$$\sum_{n \geq 0} Q_n(x,t)\, u^n = \frac{(1-t)H(u)}{H(tu) - tH(u)}$$

$\Longrightarrow$ $Q_n(x,t)$ is symmetric

To prove this we used alternative characterizations

Banner characterization

A banner is a word over

barred alphabet $\{1, \bar{1}, 2, \bar{2}, \ldots\}$

such that

- each unbarred letter is followed by a letter $\geq$ in value or is last
- each barred letter is followed by a letter $\leq$ in value

$$b = (2\ 3\ \bar{8}\ 3\ 3\ \bar{5}\ \bar{5}\ \bar{4}\ 4\ 6)$$

$$x_b = x_2\, x_3\, x_8\, x_3\, x_3\, x_5\, x_5\, x_4\, x_4\, x_6$$

<u>Th</u> $\quad Q_n(x,t) = \sum_{b \in \mathcal{B}_n} x_b\, t^{\#\mathrm{bars}(b)}$

We use this to obtain a recurrence relation for $Q_n(x,t)$

$\Rightarrow$ generating function formula

<u>P-partitions</u>

Given a finite poset P, a function $f : P \to \mathbb{Z}_{\geq 0}$ is a <u>P-partition</u> if it is weakly decreasing, that is

$f(a) \geq f(b)$ if $a < b$ in P

$f : P \to \mathbb{Z}_{>0}$ is a <u>strict</u> P-partition, if it is strictly decreasing that is

$f(a) > f(b)$ if $a < b$ in P

Let $\tilde{\mathcal{F}}_P$ be set of P-partitions

$\tilde{\tilde{\mathcal{F}}}_P$ " " " strict P-partition

Define
$$\left.\begin{aligned} K_P(x) &= \sum_{f \in \tilde{\mathcal{F}}_P} x_f \\ \tilde{K}_P(x) &= \sum_{f \in \tilde{\tilde{\mathcal{F}}}_P} x_f \end{aligned}\right\} \in QSym_n \quad n = |P|$$

<u>Th</u> (Stanley - P-partion reciplocity)

$\forall$ posets P of size n

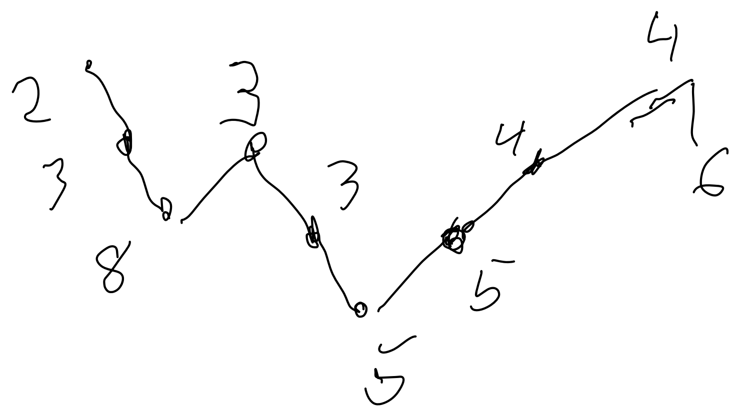
$$\omega K_P(x) = \tilde{K}_P(x)$$

ω involution on $QSym_n$

takes $F_{S,n}$ to $F_{[n-1]\setminus S,n}$

Stanley's observation

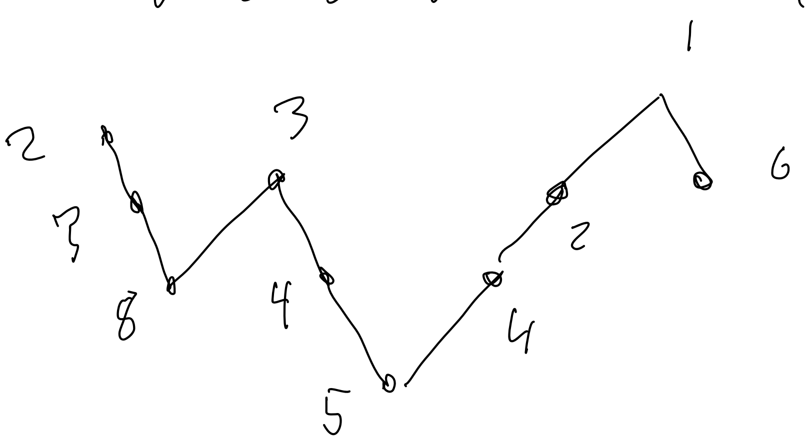
Let P be a zig-zag poset



$\in \mathcal{F}_P$

upsteps get bars

$2\;3\;\bar{8}\;3\;3\;\bar{5}\;\bar{5}\;\bar{4}\;4\;6 \in \mathcal{B}_{10}$



$\in \tilde{\mathcal{F}}_P$

2 3 8 . 3 4 5 . 4 . 2 . 1 6

Smirnov word
adjacent letters different

Descents correspond to upsteps of zigzag poset

Let SW_n be the set of Smirnov words on $\mathbb{Z}_{>0}$ - length n

$$\sum_{b \in B_n} x_b t^{\#bars(b)} = \sum_{P \in ZZ_n} K_P(x) t^{upsteps}$$

$$\sum_{w \in SW_n} x_w t^{des(w)} = \sum_{P \in ZZ_n} \tilde{K}_P(x) t^{upsteps}$$

By P-partition reciprocity

$$\omega Q_n(x,t) = \omega \sum_{b \in B_n} x_b t^{\#bars(b)}$$

$$= \sum_{w \in SW_n} x_w t^{des(w)}$$

$$=: SW_n(x,t)$$

gen function result is now

$$\sum_{n\geq 0} SW_n(x,t)\, u^n = \frac{(1-t)\,E(u)}{E(ut) - t\,E(u)}$$

Smirnov words are same as proper colorings of path graph n nodes

$w_1 - w_2 - w_3 - w_4 \quad \cdots \quad - w_n$

$SW_n(x,1)$ is chromatic symmetric function of path

Labeled path graph

$P_n = (1) - (2) \cdots - (n)$

$$SW_n(x,t) = \sum_{c \in C(P_n)} t^{des(c)} x_c$$