Recall from last time:

$$A_n(q,t) \xrightarrow{\text{lift}} Q_n(x,t) \xrightarrow{\omega} SW_n(x,t)$$

$$SW_n(x,t) = \sum_{w \in SW_n} x_w \, t^{\operatorname{des}(w)}$$

SW_n ← words over $\mathbb{Z}_{>0}$ with no adjacent repeats length n

$$= \sum_{c \in C(P_n)} x_c \, t^{\operatorname{des}(c)}$$

$$= \sum_{c \in C(P_n)} x_c \, t^{\operatorname{asc}(c)}$$

$$P_n = (1) - (2) - \cdots - (n)$$

$$X_G(x,t) = \sum_{c \in C(G)} t^{\operatorname{asc}(c)} x_c \in \mathrm{QSym}_n^{\mathbb{Q}[t]}$$

$$\operatorname{asc}(c) = |\{ ij \in E(G) \mid i<j \quad c(i)<c(j) \}|$$

$$\operatorname{des}(c) = |\{ ij \in E(G) \mid i<j \quad c(i)>c(j) \}|$$

Prop If $X_G(x,t)$ is symmetric then $t^{|E(G)|} X_G(x, t^{-1})$

$$= X_G(x,t)$$

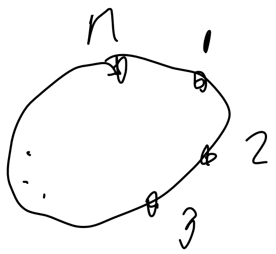
Consequently $X_G(x,t)$ is palindromic and asc(c) can be replaced by des(c).

When is $X_G(x,t)$ symmetric?

Ans Whenever G is

- natural unit interval graph (inc(P), P is a natural unit interval order) (Shareshian & W.)

- naturally labeled cycle



C_n (Ellzey & W.)

- connected components as above.

Formula from last time

$$\sum_{n\geq 0} X_{P_n}(x,t)\, u^n = \frac{(t-1)\,E(u)}{E(tu) - t\,E(u)}$$

has the consequences for X_{P_n}

- e-positive & e-unimodal

  i.e for $0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor$

  coef t^{j+1} − coef t^j is e-positive

- Frobenius char of rep of S_n on cohomology of toric variety assoc with dual permutahedron is $\omega X_{P_n}(x,t)$

  (Procesi & Stanley)

Conjecture 1 (refinement of Stanley-Stem)

If G is a natural unit interval graph then $X_G(x,t)$ is e-positive and e-unimodal

<u>Conjecture 2</u> Connection with Hessenberg Varieties (now proved by Brosnan-Chow and Guay-Paquet)

Consequences ~~of Conjecture 1~~

1. Schur-positivity Conjecture 1
   P-tableaux

2. Schur-unimodality
   This follows from Conjecture 2 and the Hard Lefschetz Th

<u>Problem</u> Find a combinatorial proof of Schur-unimodality that involves P-tableaux

3. p-positivity and p-unimodality of $\omega X(w,t)$

power-sum symmetric fn

__Th__ (Stanley 1995) For all graphs G, $X_G(x,1) = \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi)\, p_{type(\pi)}$

bond lattice of G

Consequently $\omega X_G(x,1)$ is p-positive

Our approach to p-positivity of $\omega X_G(x,t)$

Let P be a poset on $[n]$

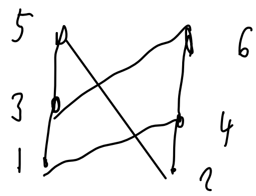
A word $a_1 \cdots a_k$ over alphabet $[n]$

- has a P-descent at i if

$$a_i >_P a_{i+1}$$

- has a left-to-right P-max at $i>1$ if $a_i >_P a_j \quad \forall j < i$

Let $\mathcal{N}_P$ be the set of all $\sigma \in \mathfrak{S}_n$, such that σ has no P-descents and no left-to-right P-max

Ex $P =$ 

$\sigma = 2\ 1\ 5\ 4\ 3\ 6 \notin \mathcal{N}_P$ no

$\sigma = 4\ 3\ \boxed{5\ 2}\ 1\ 6 \notin \mathcal{N}_P$

$$\sigma = 4\,3\,2\,1\,5\,6 \in \mathcal{N}_P$$

Th (Sh + W) Let $G = inc(P)$ where P is a nat unit int order. Then

coeff of $\frac{1}{n}p_{(n)}$ in the power-sum expansion of $\omega X_{inc(P)}(x,t)$ is

$$\sum_{\sigma \in \mathcal{N}_P} t^{inv_G(\sigma)}$$

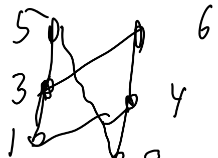
where $inv_G(\sigma) = \left| \{ \cancel{\{i,j\}}\ \{\sigma(i),\sigma(j)\} \in E(G) \mid i<j,\ \sigma(i) > \sigma(j) \} \right|$

coeff of $z_\lambda^{-1} p_\lambda$ '

Given a poset P on $[n]$ and $\lambda \vdash n$, let $\mathcal{N}_{P,\lambda}$ be the set of fillings of Young diagram λ with $1, 2, \ldots, n$, used once each, so that the rows have no

- P-descent
- left-to-right P-max

If $\ell(\lambda) = 1$ $\quad \mathcal{N}_{P,\lambda} = \mathcal{N}_P$

Ex $P =$ 

$\lambda = (3,2,1) \vdash 6$

$$\begin{matrix} 4 & 3 & 2 \\ 5 & 6 & \\ 1 & & \end{matrix} \in \mathcal{N}_{P,\lambda}$$

Conjecture (Sh & W) / Thm (Athanasiadis)

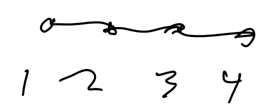
Let $G = \mathrm{inc}(P)$, P nat unit int order, and $\lambda \vdash n$. Then

coef $z_\lambda^{-1} p_\lambda$ in power-sum expansion of $\omega X_G(x,t)$ is

$$\sum_{T \in \mathcal{N}_{P,\lambda}} t^{\mathrm{inv}_G(rw(T))}$$

$rw(T)$ is perm in S_n obtained by reading T in row order

$$rw\begin{pmatrix} 4\,3\,2 \\ 5\,6 \\ 1 \end{pmatrix} = 432561$$

Ex $P =$  $G =$ 

$\lambda = (2,2)$

$$\mathcal{N}_{P,\lambda} = \left\{ \begin{array}{cccccccc} \begin{smallmatrix}1&2\\3&4\end{smallmatrix} & \begin{smallmatrix}1&2\\4&3\end{smallmatrix} & \begin{smallmatrix}2&1\\3&4\end{smallmatrix} & \begin{smallmatrix}2&1\\4&3\end{smallmatrix} & \begin{smallmatrix}3&4\\1&2\end{smallmatrix} & \begin{smallmatrix}4&3\\1&2\end{smallmatrix} & \begin{smallmatrix}3&4\\2&1\end{smallmatrix} & \begin{smallmatrix}4&3\\2&1\end{smallmatrix} \end{array} \right\}$$

$$\begin{array}{cccccccc} 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{array}$$

G - inversion

$$\sum_{T \in \eta_{\rho,\lambda}} t^{\mathrm{inv}_G(\mathrm{rw}(T))} = 1 + 3t^1 + 3t^2 + t^3$$

Note this is <u>palindromic & unimodal</u>

ρ-unimodality of $\omega X_G(x,t)$

implies $\sum_{T \in \eta_{\rho,\lambda}} t^{\mathrm{inv}_G(\mathrm{rw}(T))}$ is unimodal

This is still open.

Athanasiadis' proof - next time