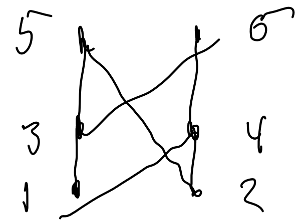
<u>Conjecture (Shareshian & W) / Theorem (Athanasiadis)</u>

$G = \mathrm{inc}(P)$, $P$ natural unit int order. Let $\lambda \vdash n$. Then coef $c_\lambda^P$ of $z_\lambda^{-1} p_\lambda$ in powersum expansion of $\omega X_G(x,t)$ is

$$\sum_{T \in \mathcal{N}_{P,\lambda}} t^{\mathrm{inv}_G(rw(T))} := \mathcal{N}_{P,\lambda}(t)$$

<u>Athanasiadis' Proof</u> - Ingredients

1. Our observation relating $\mathcal{N}_{P,\lambda}(t)$ to $\mathcal{N}_P(t)$
2. Our special case $\ell(\lambda) = 1$
3. Our decomposition in basis of fundamental quasisymmetric functions
4. A lemma implicit in work of Roichman - transfers fundamental decomp. to power-sum decomp.

① $P =$  $\lambda = (3,2,1) \vdash 6$

$T = \begin{matrix} 4 & 3 & 2 \\ 5 & 6 & \\ 1 & & \end{matrix} \in \mathcal{N}_{P,\lambda}$

$G$ any graph on $[6]$

$\mathrm{inv}_G(\mathrm{rw}(T)) = \mathrm{inv}_G(432) + \mathrm{inv}_G(56) + \mathrm{inv}_G(1)$

$+ \mathrm{inv}_G(\{2,3,4\}, \{5,6\}, \{1\})$

Given $T \in \mathcal{N}_{P,\lambda}$ let $B(T)$ be the set composition formed by the rows of $T$

$$\mathrm{inv}_G(\mathrm{rw}(T)) = \mathrm{inv}_G(B(T)) + \sum_{i=1}^{k} \mathrm{inv}_G(\mathrm{row}_i(T))$$

Follows that

$$\sum_{T \in \mathcal{N}_{P,\lambda}} t^{\mathrm{inv}_G(T)} = \sum_{B=(B_1,\dots,B_k) \in SC_\lambda} t^{\mathrm{inv}_G(B)} \prod_{i=1}^{k} \sum_{\alpha \in \mathcal{N}_{P_i}} t^{\mathrm{inv}_G(\alpha)}$$

where $P_i = P|_{B_i}$

(2) Our special case:

$\sum_{\alpha \in \eta_{P_i}} t^{\mathrm{inv}_G(\alpha)}$ is coef $c^{P_i}_{(\lambda_i)}$ of

$\frac{1}{\lambda_i} p_{(\lambda_i)}$ in $X_{\mathrm{inc}(P_i)}(x,t)$

So $\eta_{P,\lambda}(t) = \sum_{B \in SC_\lambda} t^{\mathrm{inv}_C(B)} \prod_{i=1}^{k} c^{P_i}_{(\lambda_i)}$

Want to show

$$c^P_\lambda = \sum_{B \in SC_\lambda} t^{\mathrm{inv}_C(B)} \prod_{i=1}^{k} c^{P_i}_{(\lambda_i)}$$

(3) Our decomposition into fundamentals

Th (Shareshian & W, $t=1$ Chow)

Let $P$ be a poset on $[n]$ and $G = \mathrm{inc}(P)$. Then

$$\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_G(\sigma)} F_{\mathrm{DES}_P(\sigma)}$$

## (4) Roichman's Lemma

Let $g(x) \in \mathrm{Sym}_n^{R}$ where $R$ is a commutative $\mathbb{Q}$-algebra and let $g(x) = \sum_{S \subseteq [n-1]} \alpha_S F_{S,n}$

where $\alpha_S \in R$. Then the coefficient of $z_\lambda^{-1} p_\lambda$ in the power-sum expansion of $g(x)$ is $\sum_{S \in U_\lambda} (-1)^{|S - S(\lambda)|} \alpha_S$ where

- $S(\lambda) = \{\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1}\}$
- $U_\lambda$ is the set of $\lambda$-unimodal subsets of $[n-1]$

Ex of $\lambda$-unimodal set

$\lambda = (4,3,3) \vdash 10$ $\qquad$ $S(\lambda) = \{4, 7\}$

$1 \quad 2 \quad 3 \quad \boxed{4} \quad 5 \quad 6 \quad \boxed{7} \quad 8 \quad 9$ $\qquad$ any subset of $S(\lambda)$

initial segment

$\{1\ 2\ 5\ 6\ \boxed{7}\} \in U_\lambda$

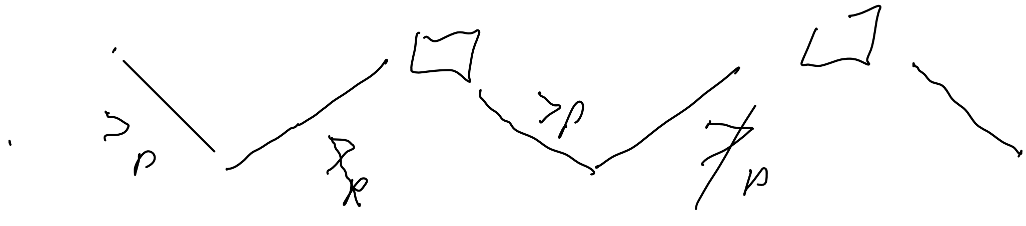
$$\omega X_{inc(P)}(x,t) = \sum_{\sigma \in S_n} F_{DES_P(\sigma)}\, t^{inv_G(\sigma)}$$

$$\alpha_S = \sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) = S}} t^{inv_G(\sigma)}$$

$$c_\lambda^P = \sum_{S \subseteq U_\lambda} (-1)^{|S - S(\lambda)|} \sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) = S}} t^{inv_G(\sigma)}$$

$$= \sum_{\substack{\sigma \in S_n \\ DES_P(\sigma) \in U_\lambda}} (-1)^{|DES_P(\sigma) - S(\lambda)|}\, t^{inv_G(\sigma)}$$

what does it mean for $DES_P(\sigma) \in U_\lambda$




segments of $\sigma$ are $P$-unimodal

Ex $\lambda = (4,3,3) \vdash 10 \qquad S(\lambda) = \{4,7\}$

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & \boxed{4} & 5 & 6 & \boxed{7} & 8 & 9 & 10 \\ 8 & 5 & 3 & \boxed{10} & 6 & 4 & \boxed{2} & 1 & 7 & 9 \end{bmatrix}$$

assume every descent is a $P$-descent

$$T = \begin{matrix} 8 & 5 & 3 & 10 \\ 6 & 4 & 2 & \\ 1 & 7 & 9 & \end{matrix}$$

count $P$-descents in each row we get

$|DES_P(\sigma) - S(\lambda)|$

Shape is $\lambda$

Let $\mathcal{M}_\lambda^P$ be set of fillings of Young diagram of shape $\lambda$ with distinct entries in $[n]$ and with $P$-unimodal rows

$$c_\lambda^P = \sum_{\sigma \in S_n} (-1)^{|DES_P(\sigma) - S(\lambda)|} t^{inv_G(\sigma)}$$

$$= \sum_{T \in \mathcal{M}_\lambda^P} \prod_{i=1}^{k} (-1)^{DES_P(row_i)} t^{inv_G(\sigma)}$$

$$= \sum_{B \in SC_\lambda} t^{inv_G(B)} \prod_{i=1}^{k} \sum_{\sigma \in S_{\lambda_i}} (-1)^{|DES_P(\sigma)|} t^{inv_G(\sigma)}$$

$\sigma$ is $P_i$-unimodal

$$= \sum_{B \in SC_\lambda} t^{\mathrm{inv}_G(B)} \prod_{i=1}^{k} c^{P_i}_{(\lambda_i)} \quad \left(\begin{array}{c}\text{Roichman}\\ \text{applied to}\\ P_i = P|_{B_i}\end{array}\right)$$

which is what we needed to prove

<u>Consequences</u> (1) Characters of $S_n$-rep on cohomology of type A semisimple Hessenberg variety are positive

(2) decomp of LLT polynomials in p-basis : Carlson & Mellit
Haglund - Wilson

Other combinatorial descriptions of $c^P_\lambda$

- Shareshian & W
- Skandera - Hyatt - ...
- Ellzey ~ Generalization to directed graphs

Th (Sh + W, Chow t=1)

$G = \mathrm{inc}(P)$ and $P$ is a poset on $[n]$.

$$\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_G(\sigma)} F_{\mathrm{Des}_P(\sigma)}$$

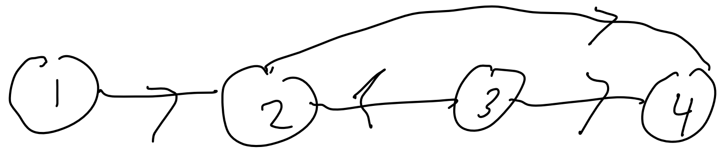
Pf (sketch) Let $AO(G)$ be set of acyclic orientations of $G$.

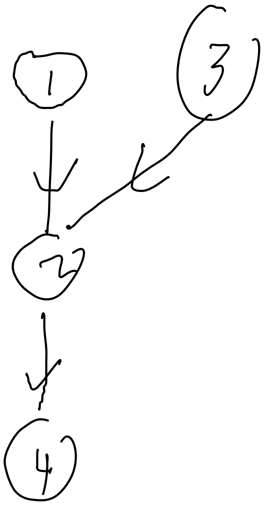
For each $\bar{a} \in AO(G)$ let $C_{\bar{a}}$ be the set of proper colorings $c$ of $G$ that are compatible with $\bar{a}$
$c(i) < c(j)$ if $i \to j$

This gives

$$X_G(x,t) = \sum_{\bar{a} \in AO(G)} t^{\mathrm{asc}(\bar{a})} \sum_{c \in C_{\bar{a}}} x_c$$

Each $\bar{a}$ determines a poset $P_{\bar{a}}$

$G$  $\bar{a}$



$P_{\bar{a}}$

Strict $P_{\bar{a}}$ - partitions ~ proper coloring compatible with $\bar{a}$

$$\tilde{K}_{P_{\bar{a}}}(x) = \sum_{c \in C_{\bar{a}}} x_c$$

Stanley's Theory of P-partition

$$\tilde{K}_P(x) = \sum_{\sigma \in \mathcal{L}(P)} F_{DES(u\sigma)}$$

where $u$ is any fixed decreasing labeling of $P$ and $\mathcal{L}(P)$ is set of linear extensions

So $$X_G(x,t) = \sum_{\bar{a} \in AO(G)} t^{asc(\bar{a})} \sum_{\sigma \in \mathcal{L}(P_{\bar{a}})} F_{DES(u_{\bar{a}}\sigma)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}(\bar{a}_\sigma)} F_{\mathrm{DES}(u_{\bar{a}_\sigma} \sigma)}$$

where $\bar{a}_\sigma$ is the unique acyclic orientation $\bar{a}$ for which $\sigma \in \mathcal{L}_{P_{\bar{a}}}$.

We show:

① $\mathrm{asc}(\bar{a}_\sigma) = \mathrm{inv}_G^a(\sigma)$

② $\exists$ a decreasing labeling $u$ of $P_{\bar{a}_\sigma}$ such that

$$\mathrm{DES}(u\sigma) = [n-1] \setminus \mathrm{DES}_P(\sigma)$$

Hence $$\omega X_G(x,t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_G(\sigma)} \mathrm{DES}_P(\sigma)$$