This lecture

1. Generalized q-Eulerian polynomials
2. Digraph version of chromatic quasisymmetric functions

(1) For each $r \in [n]$ define

$$\mathrm{inv}_{<r}(\sigma) = |\{(i,j) : 1 \le i < j \le n,\ 0 < \sigma(i) - \sigma(j) < r\}|$$

$$\mathrm{DES}_{\ge r}(\sigma) = \{i \mid \sigma(i) - \sigma(i+1) \ge r\}$$

$$\mathrm{maj}_{\ge r}(\sigma) = \sum_{i \in \mathrm{DES}_{\ge r}(\sigma)} i$$

$r = 1$ $\quad \mathrm{inv}_{<r}(\sigma) = 0 \quad \mathrm{maj}_{\ge r} = \mathrm{maj}$

$r = n$ $\quad \mathrm{inv}_{<r}(\sigma) = \mathrm{inv}(\sigma) \quad \mathrm{maj}_{\ge r}(\sigma) = 0$

Rawlings statistics $\forall r \in [n]$

$$\mathrm{inv}_{<r}(\sigma) + \mathrm{maj}_{\ge r}(\sigma)$$

r-major index

interpolates btwn inv and maj

Rawlings shows

$\mathrm{inv}_{<r}(\sigma) + \mathrm{maj}_{\geq r}(\sigma)$ is

Mahonian $\forall r$

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}_{<r}(\sigma) + \mathrm{maj}_{\geq r}(\sigma)} = [n]_q!$$

Generalization of Foata's bijection taking maj to inv

We separate the statistics

$$\sum_{\sigma \in S_n} t^{\mathrm{inv}_{<r}(\sigma)} q^{\mathrm{maj}_{\geq r}(\sigma)} := A_n^{(r)}(q,t)$$

$$A_n^{(2)}(1,t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_{<2}(\sigma)} = A_n(t)$$

<u>Th</u> (Shareshian & W)

$$A^{(2)}(q,t) = A_n(q,t)$$

$$\sum_{\sigma \in S_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{des}(\sigma)} =$$

$$= \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$\mathrm{imaj}_{\geq 2}$ is partner to des

Pf Recall

- $Q_n(x,t) = \omega X_{n\text{-path}}(x,t)$

- $ps(Q_n(x,t)) = \frac{1}{[n]_q!} A_n(q,t)$

$$Q_n(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{exc}(\sigma)} F_{DEX(\sigma)}$$

Hence

$$ps(\omega X_{n\text{-path}}(x,t)) = \frac{1}{[n]_q!} A_n(q,t)$$

Now we use

$$\omega X_G(x,t) = \sum_{\sigma \in S_n} t^{\mathrm{inv}_G(\sigma)} F_{DES_P(\sigma)}$$

where $G = \mathrm{inc}(P)$ and

$$\mathrm{inv}_G(\sigma) = \left| \left\{ \{\sigma(i), \sigma(j)\} \in E(G) \;\middle|\; \begin{array}{c} i<j \\ \sigma(i) > \sigma(j) \end{array} \right\} \right|$$

$$DES_P(\sigma) = \{ i \in [n-1] \mid \sigma(i) >_P \sigma(i+1) \}$$

$$ps(\omega X_G(x,t)) = \frac{1}{[n]_q!} \sum_{\sigma \in S_n} t^{inv_G(\sigma)} q^{\Sigma DES_P(\sigma)}$$

For $G = 1 - 2 - \cdots - n$

$$inv_G(\sigma) = inv_{<2}(\sigma)$$

$P =$ poset on $[n]$ with

$i <_P j$ if $j - i \geq 2$

So $DES_P(\sigma) = DES_{\geq 2}(\sigma)$

$$\Sigma DES_P(\sigma) = maj_{\geq 2}(\sigma)$$

Thus $ps(\omega X_{n\text{-path}}(x,t)) = \frac{1}{[n]_q!} A_n^{(2)}(q,t)$

So $A_n^{(2)}(q,t) = A_n(q,t)$,

Is there a bijective proof?

Foata's bijection taking des to exc

doesn't work

Bigeni found a bijection that does work.

We call $A_n^{(r)}(q,t)$ a generalized q-Eulerian polynomial

Recall $A_n(q,t)$ is palindromic and q-unimodal

What about $A_n^{(r)}(q,t)$?

- palindromicity - easy
- q-unimodality - not so easy

even in case $q=1$

Let $P_{n,r} = P_{(r,r+1,\dots,n,n,\dots n)}$

$$G_{n,r} = G_{(r, r+1, \ldots, n, n, \ldots, n)}$$

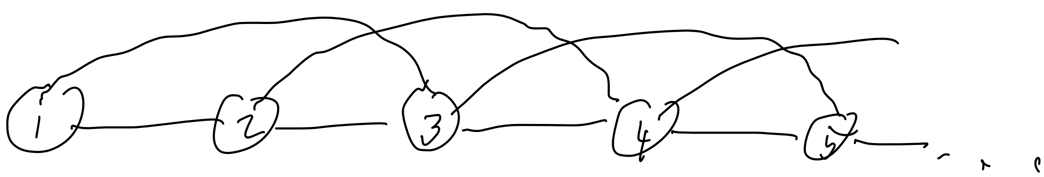
$$G_{n,r} = \mathrm{inc}(P_{n,r})$$

So $i <_{P_{n,r}} j$ iff $j - i \geq r$

and $\{i, j\} \in E(G_{n,r})$ if

$$0 < j - i < r$$

$G_{n,2}$ = n-path 

$G_{n,3}$ 

$$ps(\omega X_{G_{n,r}}) = \frac{1}{[n]_q!} A_n^{(r)}(q,t)$$

We need to know that

$X_{G_{n,r}}(x,t)$ is Schur-unimodal

$\Longrightarrow$ $A_n^{(r)}(q,t)$ is q-unimodal

Our hope was that Schur-unimodality could be obtained by generalizing the relationship btwn

$X_{G_{n,2}}(x,t)$ and the toric variety associated with the permutohedron and hard Lefschetz theorem

Stanley EC1 1.50 f [4-]

Prove $\sum_{\sigma \in S_n} t^{inv_{<r}(\sigma)}$ is unimodal

$\boxed{A_n^{(r)}(1,t)}$ DeMari & Shayman generalized Eulerian polynomial

Solution It's the Poincaré polynomial of the regular semisimple Hessenberg variety

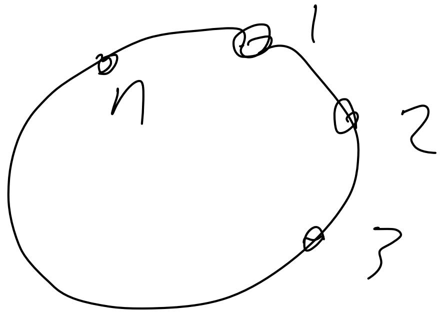
Consequently by the hard Lefschetz theorem it's unimodal.

Stanley - is there a more elementary proof?

Tymoczko representation of S_n on cohomology

$\Rightarrow$ Our conjecture relating chromatic quasisymmetric functions to Tymoczko's representation

Chromatic quasisymmetric functions for digraphs

$C_n =$ 

Stanley (1995)

$$\sum_{n\geq 2} X_{C_n}(x)\, u^n = \frac{\sum_{k\geq 2} k(k-1)\, e_k u^k}{1-\sum_{k\geq 2}(k-1)\, e_k u^k}$$

Consequently $X_{C_n}(x)$ is e-positive

Ellzey - W t - analog Smirnov word enumerators

$$\sum_{n\geq 2} X_{C_n}(x,t)\, u^n = \frac{\sum_{k\geq 2}\left([2]_t [k]_t + k\, t^2 [k-3]_t\right) e_k u^k}{1-\sum_{k\geq 2} t\, [k-1]_t\, e_k u^k}$$

Consequently $X_{C_n}(x,t)$ is e-positive

Stanley:

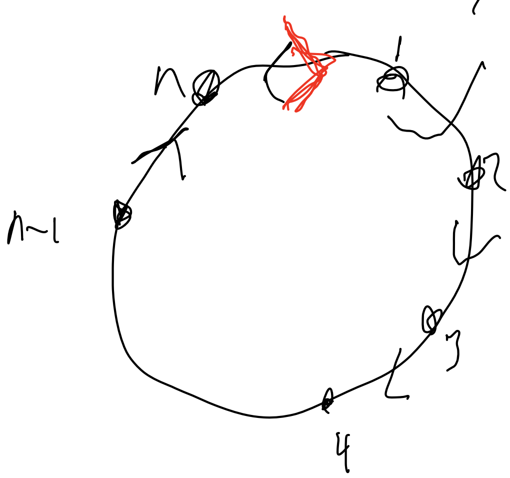
Every labeled graph can be viewed as an acyclic directed graph by orienting edges

$i — j$ as $i \to j$

if $i < j$

$\mathrm{asc}(c) = \{(i,j) \in E(\vec{G}) \mid c(i) < c(j)\}$

labeled cycle becomes



directed cycle

$\vec{C}_n$

Theorem (Ellzey)

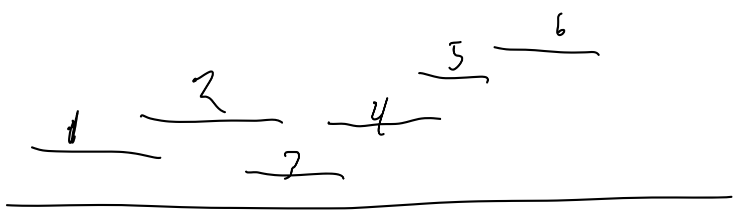
$$\sum_{n \geq 1} X_{\vec{C}_n}(x,t)\, u^n = \frac{t \sum_{k \geq 1} k\,[k-1]_t\, e_k\, u^k}{1 - t \sum_{k \geq 1} [k-1]_t\, e_k\, u^k}$$

Consequently $X_{\vec{C}_n}(x,t)$

is e-positive & e-unimodal

when is $X_{\vec{G}}(x,t)$ symmetric?

Th (Ellzey) when $\vec{G}$ is a cyclic version of natural unit interval graph, $X_{\vec{G}}(x,t)$ is symmetric

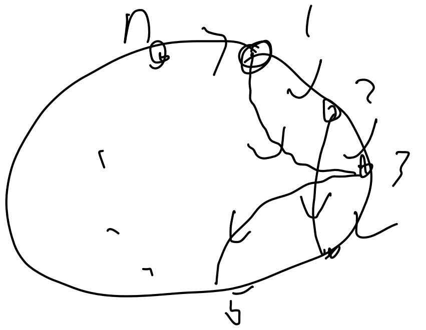


natural unit interval graph

circular indifference digraph

Examples (1) All natural unit interval graphs correspond to acyclic circular indifference digraphs

(2)



circular indifference digraph

Conjecture If $\vec{G}$ is a circular indifference digraph then $X_{\vec{G}}(x,t)$ is e-positive + e-unimodal

Th (Ellzey) $\omega X_{\vec{G}}(x,t)$ is p-positive (coeff's different since no notion of P-descents)

Open questions for digraph version

- p-unimodality
- Schur-positivity & Schur-unimodality
- Geometric interpretation