Superspace Coinvariants and Delta Operator Modules

"Vandermondes in superspace"

R. Wilson

CLASSICAL COINV RING

$$\mathfrak{S}_n \circlearrowright \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[X_n]$$

$$R_n := \mathbb{C}[X_n] \big/ \langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle = \mathbb{C}[X_n] \big/ \langle e_1, \ldots, e_n \rangle$$

(graded $\mathfrak{S}_n$-module)

CHEVALLEY $R_n \cong \mathbb{C}[\mathfrak{S}_n]$ as an ungraded $\mathfrak{S}_n$-mod

E. ARTIN $\mathrm{Hilb}(R_n; q) = [n]!_q$

LUSZTIG-STANLEY
$$\mathrm{grFrob}(R_n; q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}\, T} s_{\mathrm{shape}\, T}$$

$$= H_{(1^n)}[X; q]$$

(HALL-LITTLEWOOD)

BOREL $H^\bullet(\mathcal{F}\ell_n; \mathbb{C}) = R_n$

refln: conj to $\begin{pmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

ξ root-of-unity

finite $\mathbb{C}$-refln gp

RMK $\mathfrak{S}_n \rightsquigarrow G \subseteq GL_n\mathbb{C}$

$G \circlearrowright \mathbb{C}[X_n]$

CHEVALLEY
$$\begin{cases} \mathbb{C}[X_n]^G = \mathbb{C}[\underbrace{f_1, \ldots, f_n}_{\text{fundamental invariants}}] \\ \mathbb{C}[X_n] \big/ \langle \mathbb{C}[X_n]_+^G \rangle \cong \mathbb{C}[G] \end{cases}$$

DIAGONAL COINV

$\mathfrak{S}_n \circlearrowright \mathbb{C}[X_n, Y_n]$ $\quad w \cdot x_i = x_{w(i)}$, $w \cdot y_i = y_{w(i)}$

$$DR_n = \mathbb{C}[X_n,Y_n] \Big/ \left\langle \mathbb{C}[X_n,Y_n]_+^{\mathfrak{S}_n} \right\rangle \overset{\text{Weyl}}{=} \mathbb{C}[X_n,Y_n] \Big/ \left\langle \sum_{i=1}^n x_i^a y_i^b : a+b>0 \right\rangle$$

DOUBLY-GRADED $\mathfrak{S}_n$-module

HAIMAN ① As ungraded $\mathfrak{S}_n$-mods,

$$DR_n \cong \mathbb{C}[\underbrace{\mathrm{Park}_n}_{\text{size } n \text{ parking fcns}}] \otimes \mathrm{sign}$$

② $\mathrm{grFrob}(DR_n; \underset{x\text{-grading}}{q}, \underset{y\text{-grading}}{t}) = \nabla e_n$ OPEN Schur expansion?

relies on Haiman, uses shuffle THM

CARLSSON-MELLIT

CARLSSON-OBLOMKOV

uses geometry of affine Springer fibers

$$\mathrm{Hilb}(DR_n; q,t) = \sum_{p \in \mathrm{Park}_n} q^{\mathrm{area}(p)} t^{\mathrm{dinv}(p)}$$

SUPERSYMMETRY

<u>SUPERSPACE</u> RING

BOSONS FERMIONS

$$\Omega_n = \mathbb{C}[x_1,\dots,x_n] \otimes \Lambda\{\theta_1,\dots,\theta_n\} = \mathbb{C}[X_n; \Theta_n]$$

$x_i x_j = x_j x_i \qquad x_i\theta_j = \theta_j x_i \qquad \theta_i\theta_j = -\theta_j\theta_i$

differential forms

$\mathfrak{S}_n \circlearrowright \Omega_n \qquad w\cdot x_i = x_{w(i)}$

$\qquad\qquad\qquad w\cdot\theta_i = \theta_{w(i)}$

$SR_n = \Omega_n / \langle \Omega_+^{\mathfrak{S}_n} \rangle$ SUPERSPACE COINVARIANT RING

Bigraded $\mathfrak{S}_n$-module

<u>SOLOMON</u> What does Ω_n^G look like for G a $\mathbb{C}$-refl^n gp?

THM (Solomon) Let $f_1,\dots,f_n \in \mathbb{C}[X_n]^G$ be a system of invariants. ($\mathbb{C}[X_n]^G = \mathbb{C}[f_1,\dots,f_n]$) Then Ω_n^G is a <u>free</u> $\mathbb{C}[X_n]^G$-module w/ basis $\{df_{i_1}\cdots df_{i_k} : 1\le i_1<\dots<i_k\le n\}$

and $df = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\theta_i$ is the <u>Euler derivation</u>.

$\ulcorner \Rightarrow SR_n = \Omega_n / \langle e_1,\dots,e_n, de_1,\dots,de_n\rangle \lrcorner$

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<u>COR</u> [Shephard-Todd (CASE-BY-CASE) ; Solomon (UNIFORM)]

$$\sum_{g \in G} t^{\dim \mathrm{Fix}(g)} = (t + d_1 - 1) \cdots (t + d_n - 1)$$

$\mathrm{Fix}(g) = \{ v \in \mathbb{C}^n : gv = v \}$

where $d_i = \deg f_i$ are the FUNDAMENTAL DEGREES.

$G = \mathfrak{S}_n$

$$\sum_{k=1}^n c(n,k) \cdot t^k = t(t+1) \cdots (t+n-1).$$

$c(n,k)$ = # of $w \in \mathfrak{S}_n$ w/ k cycles

STIR # of 1ST KIND

$$SR_n = \Omega_n / \langle \Omega_+^{\mathfrak{S}_n} \rangle$$

[N. Bergeron-Zabrocki-Machacek-...]

<u>CONJ</u> (Fields Inst. Combin. Gp)

① As ungraded $\mathfrak{S}_n$-mod,

$$SR_n \cong \mathbb{C}[\mathcal{OP}_n] \otimes \text{sign}$$

ordered set ptns $(B_1|\cdots|B_k)$ of $[n]$

② $$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} [k]!_q \underbrace{S_q(n,k)}_{q\text{-stirling \# of 2ND KIND}}$$

x-deg θ-deg

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q \cdot S_q(n-1,k)$$

$$S_q(0,k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

③ $$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=0}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n \Big|_{t \to 0}$$

<u>PROGRESS</u> ② $\mathrm{Hilb}(SR_n; q, z) \geq \sum_{k=0}^{n} z^{n-k} [k]!_q \underbrace{S_q(n,k)}_{q\text{-stirling \# of 2ND KIND}}$ (Fields gp)

x-deg θ-deg

Also, $\mathrm{Hilb}(SR_n^{\text{sign}}; q, z)$ is CORRECT (SWANSON-WALLACH)