Superspace Coinvariants and Delta Operator Modules

"Vandermondes in superspace"

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LAST TIME

COINVARIANT RINGS

$$R_n = \mathbb{C}[X_n] / \langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle$$

$\to \cong \mathbb{C}[\mathfrak{S}_n]$ Chevalley

$\to \mathrm{Hilb} = [n]!_q$ E. Artin

$\to \mathrm{grFrob}(-;q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}\,T} s_{\mathrm{shape}\,T}$

Lusztig - Stanley

$$DR_n = \mathbb{C}[X_n, Y_n] / \langle \mathbb{C}[X_n, Y_n]_+^{\mathfrak{S}_n} \rangle$$

$\to \cong \mathbb{C}[\mathrm{Park}_n] \otimes \mathrm{sign}$ Haiman

$\to \mathrm{Hilb} = \sum_{\pi \in \mathrm{Park}_n} q^{\mathrm{area}} t^{\mathrm{dinv}}$ Carlsson-Mellit ; Carlsson-Oblomkov

$\to \mathrm{grFrob}(-;q,t) = \nabla e_n$ Haiman

$\mathbb{C}[x_1, \dots, x_n] \otimes \Lambda\{\theta_1, \dots, \theta_n\}$

$$SR_n = \Omega_n / \langle \Omega_+^{\mathfrak{S}_n} \rangle$$

$\xrightarrow{\text{Fields CONJ.}} \cong \mathbb{C}[\mathcal{OP}_n] \otimes \mathrm{sign}$

$\xrightarrow{\text{Fields CONJ.}} \mathrm{Hilb} = \sum_k z^{n-k} [k]!_q \, S_q(n,k)$

$\xrightarrow{\text{Fields CONJ}} \mathrm{grFrob}(-;q,z) = \sum_k z^{n-k} \Delta'_{e_{k-1}} e_n \big|_{t \to 0}$

SWANSON-WALLACH:

$\mathrm{Hilb}(SR_n^{\mathrm{sign}}; q, z)$ CORRECT

SAGAN-SWANSON:

CONJECTURAL "ARTIN" BASIS of SR_n

<u>FACT</u> We have an action $\odot : \mathbb{C}[X_n] \times \mathbb{C}[X_n] \to \mathbb{C}[X_n]$

$$f \odot g := (\partial \bar{f})(g)$$

$f(x_1, \dots, x_n)$

$\bar{f}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$

$\mathbb{C}$ CONJUGATE COEFFS of f

This induces a NONDEG ~~(POSITIVE DEF)~~ ~~BILINEAR FORM~~ INNER PRODUCT on $\mathbb{C}[X_n]$

$$\langle f, g \rangle := \text{CONSTANT TERM of } f \odot g.$$

<u>DEF</u> Let $I \subseteq \mathbb{C}[X_n]$ be any homogeneous ideal.
The <u>harmonic space</u> $I^\perp \subseteq \mathbb{C}[X_n]$ is

$$I^\perp = \{ g \in \mathbb{C}[X_n] : f \odot g = 0 \ \forall f \in I \}.$$

$$\Rightarrow \mathbb{C}[X_n] = I \oplus I^\perp \Rightarrow \mathbb{C}[X_n]/I \cong I^\perp$$

RING STRUCT. ☺

ONLY A VECTOR SPACE ☹

NEED TO WORK w/ COSETS $f+I$ ☹

HONEST POLYNOMIALS! ☺

FACT If $I \subseteq \mathbb{C}[X_n]$ is $I = \langle \mathbb{C}[X_n]_+^{\mathfrak{S}_n} \rangle$

$$= \langle e_1, e_2, \ldots, e_n \rangle,$$

then $I^\perp = \{ f \in \mathbb{C}[X_n] : e_d \odot f = 0 \text{ for } d = 1, 2, \ldots, n \}$

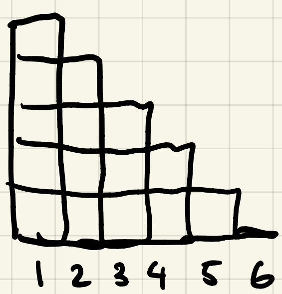
$$V_n = \begin{cases} = \text{smallest subspace of } \mathbb{C}[X_n] \text{ containing} \\ \delta_n = \underset{\uparrow}{\varepsilon_n} \cdot \left(x_1^{n-1} \cdots x_{n-1}^{1} \cdot x_n^{0} \right) \\ \quad \sum_{w \in \mathfrak{S}_n} (\operatorname{sign} w) \cdot w \\ \text{and closed under } \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}. \end{cases}$$

<u>Pf</u> First show $\dim V_n \geq n!$ (look at substaircase monomials)

Then show $e_d \odot \delta_n = 0$ for $d = 1, 2, \ldots, n$.

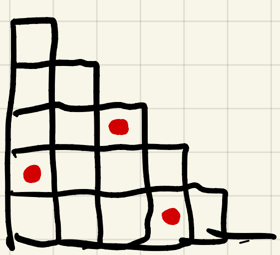
symm. diff op.

$$e_d \odot \delta_n = e_d \odot \left(\varepsilon_n \cdot \right.$$



$$\left. \right)$$

$$= \varepsilon_n \cdot \left(e_d \odot \right.$$



$$\left. \right)$$

(d = 3)

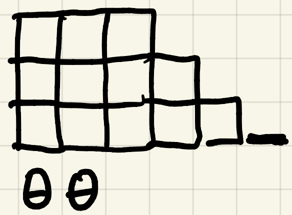
$$\frac{\partial}{\partial x}(x^3) = 3x^2$$

$$= 0.$$

Def (R-Wilson) Let $k \leq n$. The superspace Vandermonde $\delta_{nk} \in \Omega_n$ is

$$\delta_{nk} = \varepsilon_n \cdot \left(x_1^{k-1} \cdots x_{n-k}^{k-1} \, x_{n-k+1}^{k-1} \, x_{n-k+2}^{k-2} \cdots x_{n-1}^{1} \, x_n^{0} \;\; \theta_1 \cdots \theta_{n-k} \right).$$

$\ulcorner \delta_{n,n} = \delta_n \lrcorner$

e.g. $\delta_{6,4} = \varepsilon_6 \Bigg($  $\Bigg)$

graded $\mathfrak{S}_n$-mod

Def Let $V_{n,k} \subseteq \Omega_n$ be the smallest linear subspace st

- $\delta_{n,k} \in V_{n,k}$
- $V_{n,k}$ is closed under $\frac{\partial}{\partial x_1}, -, \frac{\partial}{\partial x_n}$

THM (R-Wilson) $\mathrm{grFrob}(V_{n,k}; q) = \Delta'_{e_{k-1}} e_n \Big|_{t \to 0}$

BRIDGE CONJ The map $\bigoplus_k V_{n,k} \hookrightarrow \Omega_n \twoheadrightarrow SR_n$ is a bij'n.

($\Rightarrow$ Fields CONJ)

THM (R-Wilson) $\operatorname{grFrob}(V_{n,k};q) = \Delta'_{e_{k-1}} e_n \big|_{t\to 0}$

Pf (outline)

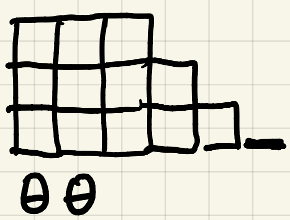
① Show $\dim V_{nk} \geq k! \cdot S(n,k)$ (look at monomials)

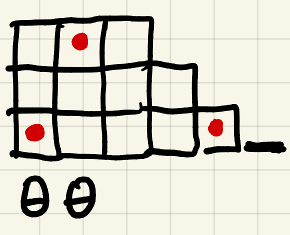
Haglund-R-Shimozono

If $R_{nk} = \mathbb{C}[X_n]/I_{nk}$ w/ $I_{nk} = \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$

then $\operatorname{grFrob}(R_{nk};q) = (\mathrm{worev}_q)\, \Delta'_{e_{k-1}} e_n \big|_{t\to 0}$.

② Show that I_{nk} annihilates δ_{nk} under $\odot$.

$e_3 \odot \delta_{6,4} = e_3 \odot \varepsilon_6 \cdot \Big($  $\Big)$

$= \varepsilon_6 \cdot \Big(e_3 \odot$  $\Big) = 0$.

at least one • appears in a non-θ column.

$d_1^{\varepsilon_1} \cdots d_{n-1}^{\varepsilon_{n-1}} \cdot (\delta_n)$

$d_j = \sum_i \frac{\partial^j}{\partial x_i^j} \theta_i \qquad \varepsilon_j = 0 \text{ or } 1$

RMK Fields group has a CONJECTURAL descr. of harmonic space to SR_n.

ZABROCKI CONJ.

bosonic, bosonic, fermionic; $X \to q$, $Y \to t$, $\Theta \to z$

$$\operatorname{grFrob}\left(\mathbb{C}[X_n, Y_n; \Theta_n] \big/ \langle \mathbb{C}[X_n, Y_n; \Theta_n]^{\mathfrak{S}_n}_+ \rangle ; q, t, z\right)$$

$$= \sum_{k=0}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n$$

CONJ (R-Wilson) Let $\mathbb{V}_{nk}$ be the smallest subsp. of $\mathbb{C}[X_n, Y_n; \Theta_n]$ st

① $\delta_{nk} \in \mathbb{V}_{nk}$ (in X_n, Θ_n)

② $\mathbb{V}_{nk}$ is closed under $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ $i = 1, \ldots, n$

③ $\mathbb{V}_{nk}$ is closed under

POLARIZATION

$$y_1 \frac{\partial^j}{\partial x_1^j} + \cdots + y_n \frac{\partial^j}{\partial x_n^j} \quad (j \geq 1)$$

Then $\operatorname{grFrob}(\mathbb{V}_{nk}; q, t) = \Delta'_{e_{k-1}} e_n$