Delta and Theta Operator Expansions

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Origins: - with Angela Hicks ($\sim$2017)

$\Delta_{m_\gamma} e_n$ or $\Delta_{s_\gamma} e_n$ at $t=1$

- my thesis
- with Alessandro Iraci

Reasons it may be interesting:

- a wonderful application for forgotten symmetric functions
- it gives an accessible method for giving e-positivities and the combinatorial side related to Macdonal polynomial operators.

Some motivation.

Conjecture (Zabrocki)

$$\mathcal{F}\left(\mathbb{C}[X_n, Y_n, \Gamma_n] \Big/ \left(\mathbb{C}[X_n, Y_n, \Gamma_n]^{S_n}_+\right)\right)$$

$$= \sum_{k=1}^{n} u^{n-k} \Delta'_{e_{k-1}} e_n$$

$\Gamma_n = \gamma_1, \dots, \gamma_n \qquad \gamma_i \gamma_j = -\gamma_j \gamma_i$

Conjecture (D'Adderio, Iraci, Vanden Wyngaerd)

$\mathcal{A} = \mathbb{C}[X_n, Y_n, \Gamma_n, T_n]$

with $\gamma_i \gamma_j = -\gamma_j \gamma_i$, $\tau_i \tau_j = -\tau_j \tau_i$.

$$\mathcal{F}\left(\mathcal{A} / (\mathcal{A}_+^{S_n}) \right) = \sum_{r,s} u^r v^s \Theta_{e_r} \Theta_{e_s} \nabla e_{n-r-s}$$

Θ-operators are important in giving the compositional Delta Conjecture

which was proved by D'Adderio, Mellit.

Our main question:

How do Delta and Theta operators behave together?

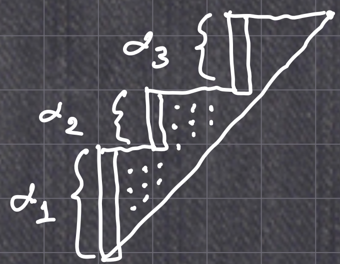
$\Delta_F \Theta_G H = ?$ F, G, H are symmetric functions

We will see things that are not Schur positive but

- Are e-positive when $t=1$
- e-positivity phenomenon

  e-positivity when $t=t$, $q=1+q$

An example: $\nabla e_n = \sum_{D \in D_n} q^{\mathrm{area}(D)} LLT_D[x;t]$



when $t=1$

$$LLT_D[X;1] = e_{\alpha(D)}$$

$$\alpha(D) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$

Theorem (D'Adderio, explicit formula proved by Alexandersson and Sulzgruber)

$LLT[X;1+q]$ is e-positive.

$$\nabla e_n \Big|_{q \to 1+q} = \sum_{D \in D_n} \sum_{S \subseteq Area(D)} q^{|S|} t^{stat(S,D)} e_{\alpha(S,D)}$$

Another example we will see later

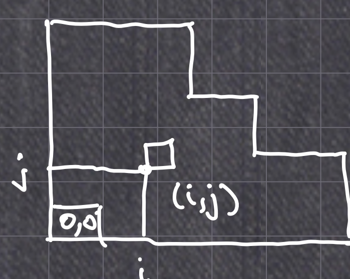
$\Delta_{m_\gamma} e_n \big|_{t=1}$ is e-positive

$\Delta_{s_\gamma} e_n$ exhibits the e-pos. phenomenon

Our goal and Definitions.

$$\Delta_F \widetilde{H}_\mu = F[B_\mu] \widetilde{H}_\mu$$

$$B_\mu = \sum_{(i,j) \in \mu} q^i t^j$$



$$\Pi \widetilde{H}_\mu = \Pi_\mu \widetilde{H}_\mu \qquad \Pi_\mu = \prod_{(i,j) \in \mu / (1)} (1 - q^i t^j)$$

$$\Theta_G = \Pi\, \underline{G}^* \,\Pi^{-1} \quad \text{with } G^* = *(G) = G\left[\tfrac{X}{M}\right]$$

$$M = (1-q)(1-t), \quad \tfrac{X}{M} \text{ "=" } (x_i q^r t^s)$$

$$B_\mu \Big|_{t=1} = \sum_i [\mu_i]_q$$

$$\sum_i [\mu_i]_q = \sum_i q^0 + q^1 + \cdots + q^{\mu_i - 1}$$

We are going to look at

$$\Delta_F\, \Theta_G\, H \quad \text{where} \quad F = m_\gamma \cdot e_1$$

$$G = e_\lambda$$

$$H = \Pi\, e_\mu^*$$

$$\Theta_G H = \Pi\, \underline{e}_\lambda^* \,\Pi^{-1} \Pi\, e_\mu^*$$

$$= \Pi\, e_\lambda^* e_\mu^*$$

It suffices to look at $\Delta_{m_\gamma} M \Delta_{e_1} \Pi * (e_\lambda)$

$$M \Delta_{e_1} \Pi * = \Xi$$

We will look at

$$\Delta_{m_\gamma} \Xi\, e_\lambda \quad \text{and} \quad \Delta_{m_\gamma} \Xi\, s_\lambda$$

at $t = 1$.

Conjecture. The following symmetric functions exhibit the e-pos. phenomenon

$$\Delta_{s_\gamma} \overset{\sim}{\sum} s_\lambda \quad \text{and therefore } \Delta_{s_\gamma} \overset{\sim}{\sum} e_\lambda$$

$$t = t \qquad q \to 1+q.$$

Symmetric Function

The combinatorial formula for the forgotten basis

For $\mu \vdash n$ with $\ell(\mu) = \ell$

$$m_\mu = \sum_{\alpha \in R(\mu)} \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

where $R(\mu)$ = set of rearrangements of μ
= compositions rearranging to μ.

Example. $\mu = (3,2,2,1,1,1)$

$\alpha = (2,1,1,3,2,1) \in R(\mu)$

$$f_\mu = (-1)^{n-\ell} \sum_{\alpha \in R(\mu)} \sum_{i_1 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

$i_1 = i_2 = 1$, $i_3 = i_4 = i_5 = 4$, $i_6 = 7$ gives the

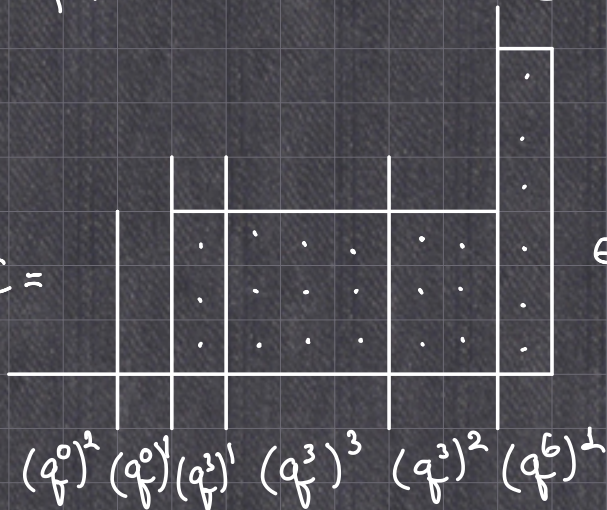
monomial $x_1^2 x_1^1 x_4^1 x_4^3 x_4^2 x_7^1$

The principal specialization

$$f_\mu\left[\frac{1}{1-q}\right] = (-1)^{n-\ell} \sum_{\alpha \in R(\mu)} \sum_{0 \le i_1 \le \cdots \le i_\ell} (q^{i_1})^{\alpha_1} (q^{i_2})^{\alpha_2} \cdots (q^{i_\ell})^{\alpha_\ell}$$

In pictures: From the example $x_1 = 1,\ x_2 = q, \ldots$

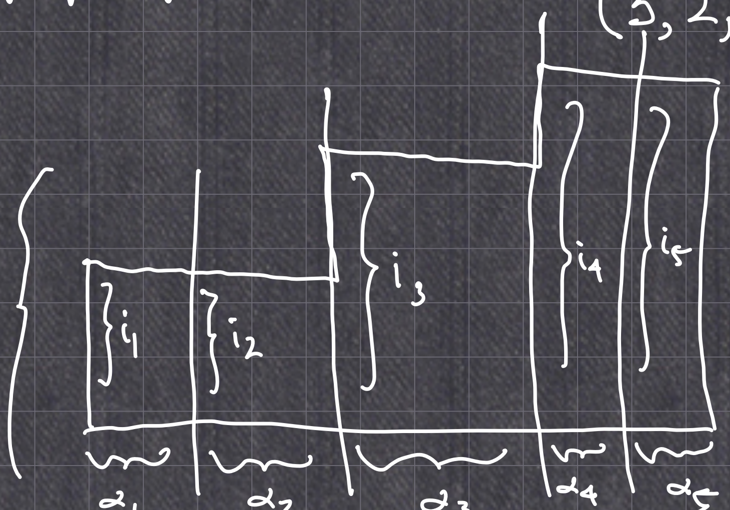
$x_i = q^{i-1}$

$C =$ 

$\in CC_\mu$

$|C| = 24$

A column-composition tableau of type $(3,2,2,1,1,1)$

$CC_\mu = \Big\{$ 

$: \alpha \in R(\mu) \Big\}$

Cauchy Identity and modified Macdonald poly.

when $t = 1$, we have $\tilde{H}_\mu[X; q, 1] = (q;q)_\mu\, h_\mu\left[\frac{X}{1-q}\right]$

$\frac{X}{1-q}$ "=" $x_1, x_1 q, x_1 q^2, \ldots, x_2, x_2 q, \ldots, x_3, \ldots$

One expansion:

Cauchy Identity

If $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$ are two homogeneous dual bases under the Hall scalar product

$$\left(\langle p_\lambda, p_\mu \rangle = z_\mu \chi(\lambda=\mu) \quad \text{or} \quad \langle s_\lambda, s_\mu \rangle = \chi(\lambda=\mu) \right)$$

then for any expressions X, Y

$$h_n[XY] = \sum_{\lambda \vdash n} u_\lambda[X]\, v_\lambda[Y]$$

$$= \sum_{\lambda \vdash n} m_\lambda[X]\, h_\lambda[Y]$$

$$= \sum_{\lambda \vdash n} s_\lambda[X]\, s_\lambda[Y]$$

$$= \sum_{\lambda \vdash n} e_\lambda[X]\, f_\lambda[Y].$$

$$h_\alpha\left[\frac{X}{1-q}\right] = \prod_{i=1}^{\ell(\alpha)} h_{\alpha_i}\left[\frac{X}{1-q}\right]$$

$$= \prod_{i=1}^{\ell(\alpha)} \sum_{\nu^i \vdash \alpha_i} e_{\nu^i}[X]\, f_{\nu^i}\left[\frac{1}{1-q}\right]$$

$$= \sum_{\eta \vdash n} e_\eta \sum_{(\nu^1, \ldots, \nu^{\ell(\alpha)}) \in PR(\eta, \alpha)} f_{\nu^1}\left[\frac{1}{1-q}\right] \cdots f_{\nu^{\ell(\alpha)}}\left[\frac{1}{1-q}\right]$$

$(\nu^1, \dots, \nu^{\ell(\alpha)}) \in PR(\eta, \alpha)$ means

$\nu^i \vdash \alpha_i$, and $\nu^1, \dots, \nu^{\ell(\alpha)}$ together rearrange to η.

$$(q;q)_\alpha h_\alpha\left[\frac{X}{1-q}\right] =$$

$$\sum_{(w^1, \dots, w^{\ell(\alpha)})} q^{\mathrm{revmaj}(w^1) + \mathrm{revmaj}(w^2) + \dots + \mathrm{revmaj}(w^{\ell(\alpha)})} x_{w^1} x_{w^2} \cdots x_{w^\ell}$$

w^i is a word in $\{1, 2, \dots\}$

$\ell(w^i) = \alpha_i$

$$= \sum_{\lambda \vdash n} m_\lambda \sum_{\vec{w} \in WV(\lambda, \alpha)} q^{\mathrm{revmaj}(\vec{w})}$$

Example: $(2\,1\,2,\ 4,\ 3\,4\,1\,3\,1) \in WV((3,2,2,2),(3,1,5))$

multiplicities of $1 : 3$

$2 : 2$

$3 : 2$

$4 : 2$

lengths: $(3, 1, 5) = \alpha$

$\Rightarrow \lambda = (3,2,2,2)$

$$\boxed{2\ 1\ 2}\quad \boxed{4}\quad \boxed{3\ 4\ 1\ 3\ 1} = \vec{w}$$

$$1 \qquad\qquad 4 \quad 2$$

$$q^{\mathrm{revmaj}(\vec{w})} = q^{1+4+2} = q^7$$

Another version

$$(q;q)_\alpha \, h_\alpha\left[\frac{X}{1-q}\right] = \sum_{\lambda \vdash n} s_\lambda \sum_{\vec{\omega} \in LWV(\lambda,\alpha)} q^{\mathrm{revmaj}(\vec{\alpha})}$$

example: $\underset{1}{1}\ \underset{2}{1}\ \underset{3}{2}\ \underset{4}{1}\ \underset{5}{3}\ 2\ 1\ 3\ 2$

$$\begin{array}{|llll}
5 & 8 & & \\
3 & 6 & 9 & \\
1 & 2 & 4 & 7 \\
\hline
\end{array}$$

$\lambda = (4,3,2)$

$\boxed{1\ 1\ 2\ 1}\quad \boxed{3}\quad \boxed{2\ 1\ 3\ 2} \in LWV((4,3,2),(4,1,4))$

$\mathrm{revmaj}(\vec{\omega}) = 2+2 = 4$