Delta and Theta operator expansions #2

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Recall that we are trying to compute

$$\Delta_{m_\gamma} M \Delta_{e_1} \Pi\, e_\lambda^* = \Delta_{m_\gamma} \Xi\, e_\lambda$$

Reminder: $\Delta_F \tilde{H}_\mu = F[B_\mu] \tilde{H}_\mu$, $\Pi \tilde{H}_\mu = \Pi_\mu \tilde{H}_\mu$.

A definition of the modified Macdonald basis:

Definitions. ($*$-scalar product)

$$\langle F, G \rangle_* = \langle F, (\omega G)[XM] \rangle$$

← Hall scalar product

$\{\tilde{H}_\mu\}_\mu$ can be defined as the basis satisfying

1. Orthogonality relation

$$\langle \tilde{H}_\mu, \tilde{H}_\lambda \rangle_* = \chi(\lambda=\mu)\, w_\mu$$

$$w_\mu = \prod_{c \in \mu} \left(q^{arm(c)} - t^{leg(c)+1}\right)\left(t^{leg(c)} - q^{arm(c)+1}\right)$$

2. Linear functional relation

$$\langle \tilde{H}_\mu, h_n \rangle = 1.$$

Now, to apply $\Delta_{m_\gamma} M \Delta_{e_1} \Pi$ to e_λ^*, we expand e_λ^* in terms of $\{\tilde{H}_\mu\}_\mu$.

$$e_\lambda^* = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{\omega_\mu} \langle \tilde{H}_\mu, e_\lambda^* \rangle_* \longrightarrow \quad e_\lambda\left[\frac{X}{M}\right]$$

$$(\omega e_\lambda)\left[\frac{X \cdot M}{M}\right]$$

$$= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{\omega_\mu} \langle \tilde{H}_\mu, h_\lambda \rangle$$

$$= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{\omega_\mu} M_{\lambda,\mu} \qquad \tilde{H}_\mu = \sum_{\lambda \vdash n} M_{\lambda,\mu} m_\lambda$$

$$e_n^* = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{\omega_\mu} \langle \tilde{H}_\mu, h_n \rangle$$

$$= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{\omega_\mu}$$

FACT:

$$e_n = \sum_{\mu \vdash n} \frac{M B_\mu \Pi_\mu}{\omega_\mu} \tilde{H}_\mu = M \Delta_{e_1} \Pi e_n^*$$

$$= \nabla e_n = e_n$$

$$\Delta_{m_\gamma} \nabla e_\lambda$$

$$= \sum \frac{M B_\mu \Pi_\mu}{\omega_\mu} M_{\lambda,\mu} m_\gamma[B_\mu] \tilde{H}_\mu$$

we want this at $t=1$.

1. $\frac{\mathcal{M} B_\mu \Pi_\mu}{\omega_\mu}\Big|_{t=1} = \frac{1}{(q;q)_\mu} (-1)^{|\mu|-\ell(\mu)} \sum_{\alpha \in R(\mu)} (1-q^{\alpha_1})$

$= \frac{1}{(q;q)_\mu} f_\mu[1-q]$

2. $\mathcal{M}_{\lambda,\mu}\Big|_{t=1} = \sum_{\vec{\omega} \in WV(\lambda,\alpha)} q^{\text{revmaj}(\vec{\omega})}$

3. $m_\gamma[B_\mu]\Big|_{t=1} = m_\gamma\left[\sum_i [\mu_i]_q\right]$

$= m_\gamma\left[\sum [\alpha_i]_q\right]$

$\alpha = (3,1,5)$ $\gamma = (4,3,2,2,1)$

q^2+q+1 1 $q^4+q^3+q^2+q+1$.

0 2 0 1 2 0 4 3 0

$(q^1)^2$ $(1)^1$ $(q^4)^2$ $(q^2)^4 (q^1)^3$

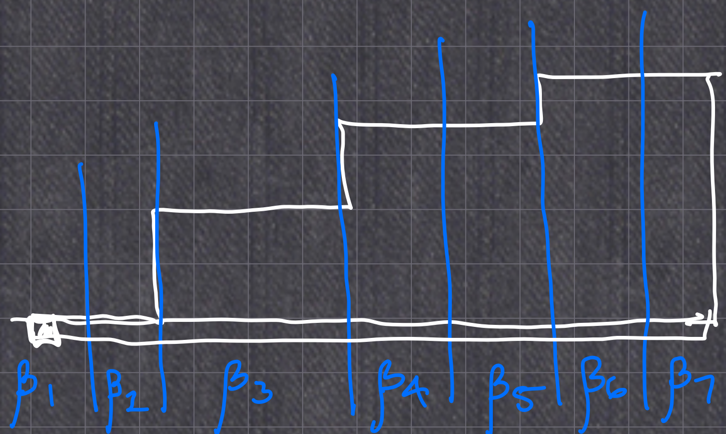
$2(1) + 1(0) + 2(4) + 4(2) + 3(1)$

$\sum i \times \#\{\text{cells to its right}\}$

4. $\tilde{H}_\mu[X;q,1] = (q;q)_\mu h_\mu\left[\frac{X}{1-q}\right]$

$$(q;q)_\mu \sum_{\eta \vdash n} e_\eta \sum_{\vec{\nu} \in PR(\eta,\alpha)} (-1)^{|\alpha|-\ell(\alpha)} CC_{\nu^1}(q) \cdots CC_{\nu^\ell}(q)$$

$$(1-q^{\alpha_1})\, CC_{\nu^1}(q) \cdots CC_{\nu^{\ell(\alpha)}}(q).$$



$$(1-q^{|\nu|})\, CC_\nu(q) = CC_\nu - \underbrace{q^{|\nu|} CC_\nu(q)}_{\sum_{\substack{C \in CC_\nu \\ c_1(C) > 0}} q^{|C|}}$$

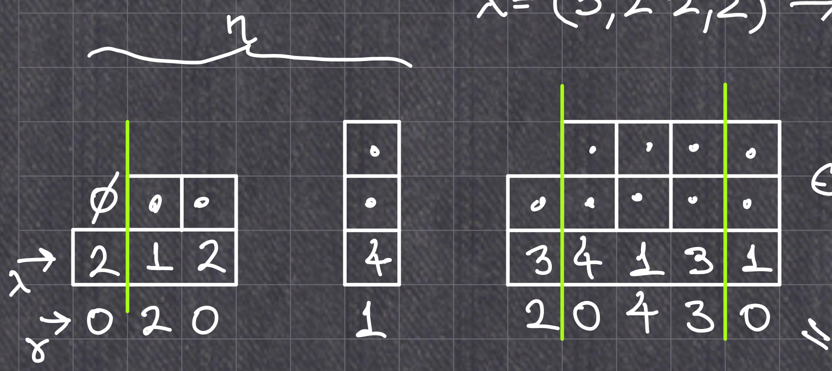
$c_1(C)$ = # cells in first column

$c_\ell(C)$ = # cells in last column

$$\Rightarrow (1-q^{|\nu|})\, CC_\nu(q) = \sum_{\substack{C \in CC_\nu \\ c_1(C) = 0}} q^{|C|} = \overline{CC}_\nu(q)$$

$\mu = (5,3,1) \to \alpha = (3,1,5)$

$\lambda = (3,2,2,2) \Rightarrow$ content 111223344



$\in LC^{\gamma}_{\lambda,\eta}$

$\nu^1 = (2,1)$

$\nu^2 = (1)$

$\nu^3 = (3,1,1)$

$\approx (T_1, T_2, T_3)$

rev maj: $1+0+4+2 = 7$

labels contribute: $2 + 2(4) + 4(2) + 3 = 21$

$\eta = (3,2,1,1,1,1)$

$\gamma = (4,3,2,2,1)$

$\lambda = (3,2,2,2)$

$$\text{weight}(T) = q^{\text{revmaj}(\bar{w}) + \text{contribution of labels} + \#\text{cells above}}$$

$$= q^{7+21+13}$$

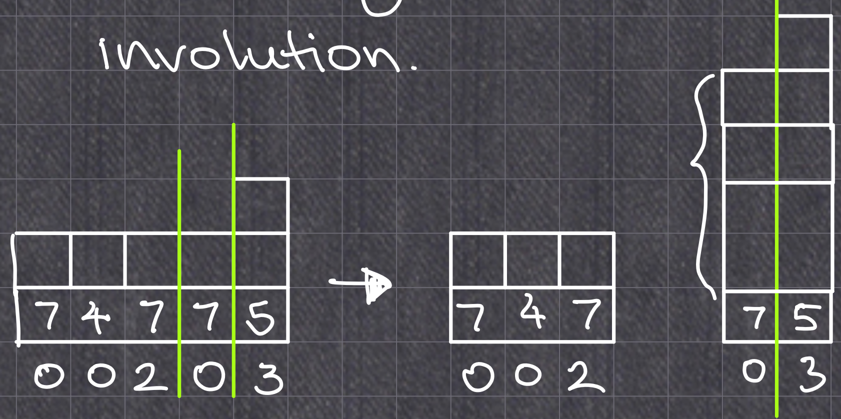
$$\text{sign}(T) = (-1)^{\#\text{vertical bars}}$$

$$= (-1)^3$$

We have shown that

$$\Delta_{m_\gamma} \Xi e_\lambda = \sum_{\eta \vdash n} e_\eta \sum_{T \in LC^{\gamma}_{\lambda,\eta}} \text{weight}(T)\,\text{sign}(T)$$

Use a weight preserving, sign-reversing involution.



$7\,4\,7\,7\,5$; $0\,0\,2\,0\,3$; $2(2)$

$7\,4\,7\,7$ → $\#\text{asc}(7477) = 1 = \text{asc}(S_1; S_2)$

$0\,0\,2$ → $0+0+2 = |L(S_1)|$; $2(0)$

If S has a vertical bar, then we can split S as $\text{split}(S) = (S_1, S_2)$

And we can join (S_1, S_2) if it came from being split.

The part on the right (S_2) was attained by adding

$$\# \operatorname{asc}(7\,4\,6\,7) + \sum \text{labels below the base } (0+0+2) = \operatorname{asc}(S_1 ; S_2) + |\ell(S_1)|.$$

The point: $\operatorname{split}(S) = (S_1, S_2)$

$$\Longleftrightarrow \quad c_1(S_2) \geq c_\ell(S_1) + \operatorname{asc}(S_1 ; S_2) + |\ell(S_1)|.$$

$\Longrightarrow$

Proposition. The following map Ψ is a sign-reversing involution on $LC^{\gamma}_{\lambda,\eta}$.

Take $T = (T_1, T_2, \ldots, T_r) \in LC^{\gamma}_{\lambda,\eta}$

Check if T_1 has a vertical bar.

If so, $\Psi(T) = (\operatorname{split}(T_1), T_2, \ldots, T_r)$

If not check if T_1 can join T_2

$$\Psi(T) = (\text{join}(T_1, T_2), T_3, \ldots, T_r)$$

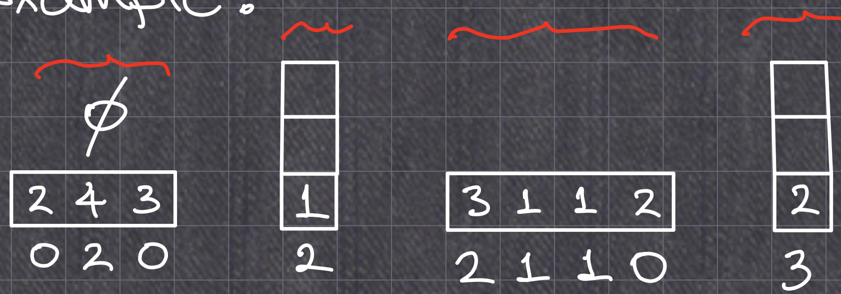
if so, otherwise apply Ψ inductively on $(T_2, \ldots, T_r)$

In the end we have fixed points $\mathcal{U}^{\gamma}_{\lambda,\eta}$ satisfying

1. No vertical bars $\Rightarrow$ 1 row
   $\Rightarrow$ $c_1(T_i) = c_2(T_i) = c(T_i)$

2. $c(S_{i+1}) < c(S_i) + \text{asc}(S_i ; S_{i+1}) + |L(S_i)|$.

Example:



$\in \mathcal{U}^{(3,2,2,2,1,1)}_{(3,3,2,1),}$ $(4,3,1,1)$

$c(S_2) < c(S_1) + \text{asc}(S_1 ; S_2) + |L(S_1)|$
$\quad\quad\quad 0 \quad + \text{asc}(2\,4\,3\,1) + 0+2+0$

$c(S_2) < 3$

$c(S_3) < 2 + asc(1\,3) + 2 = 5$

$c(S_4) < 0 + asc(3\,1\,1\,2\,2) + 2 + 1 + 1 + 0$

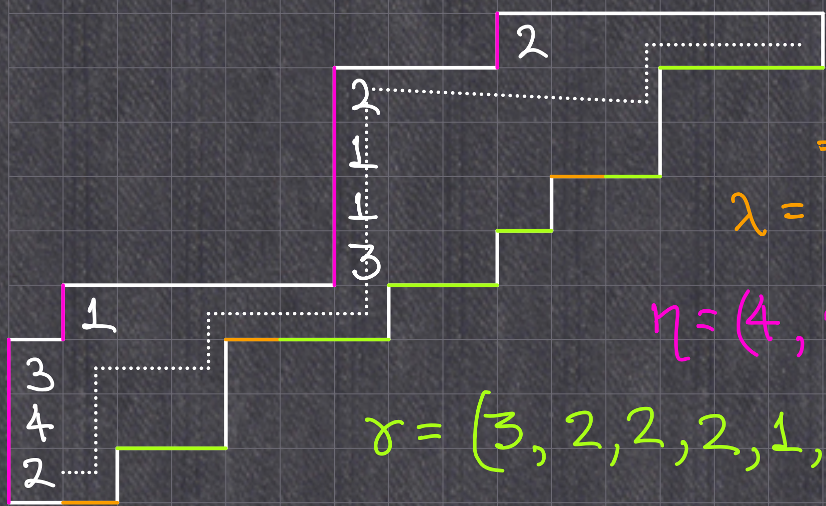
$$\Delta_{m_\gamma} \Xi e_\lambda = \sum_{\eta \vdash n} e_\eta \sum_{T \in \mathcal{U}^{\gamma}_{\lambda,\eta}} weigh(T)$$

Theres a bijection

$$\mathcal{U}^{\gamma}_{\lambda,\eta} \longleftrightarrow AP^{\gamma}_{\lambda,\eta}$$

A parallelogram polyomino

ascent compatible labels



#1's, #2's, #3's, ...

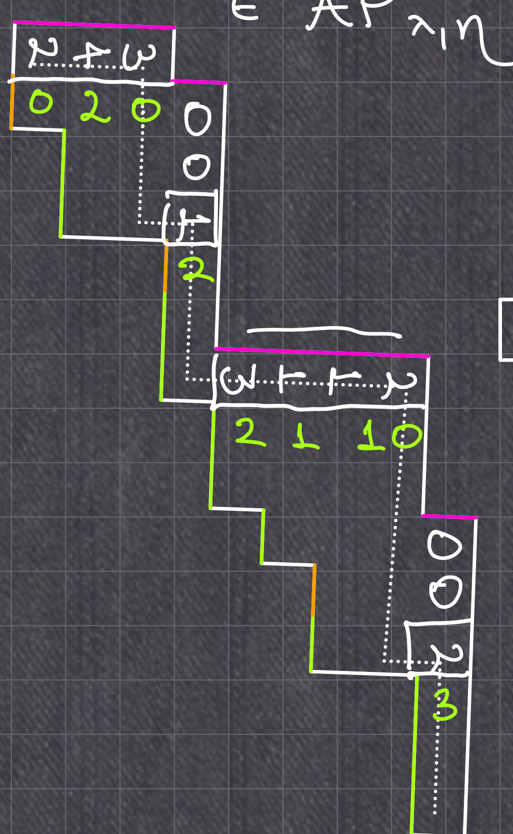
$\lambda = (3, 3, 2, 1)$

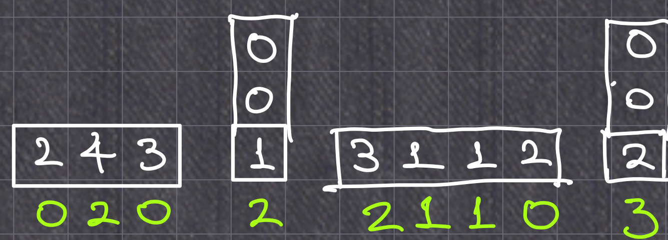
$\eta = (4, 3, 1, 1)$

$\gamma = (3, 2, 2, 2, 1, 1)$

$\in AP^{\gamma}_{\lambda, \eta}$

area = # cells between — dashed line





$$\Delta_{m_\gamma} \Xi e_\lambda = \sum_{P \in \mathcal{LP}^{\gamma}_{\lambda,\eta}} q^{\mathrm{area}(P)} e_{\eta(P)}$$

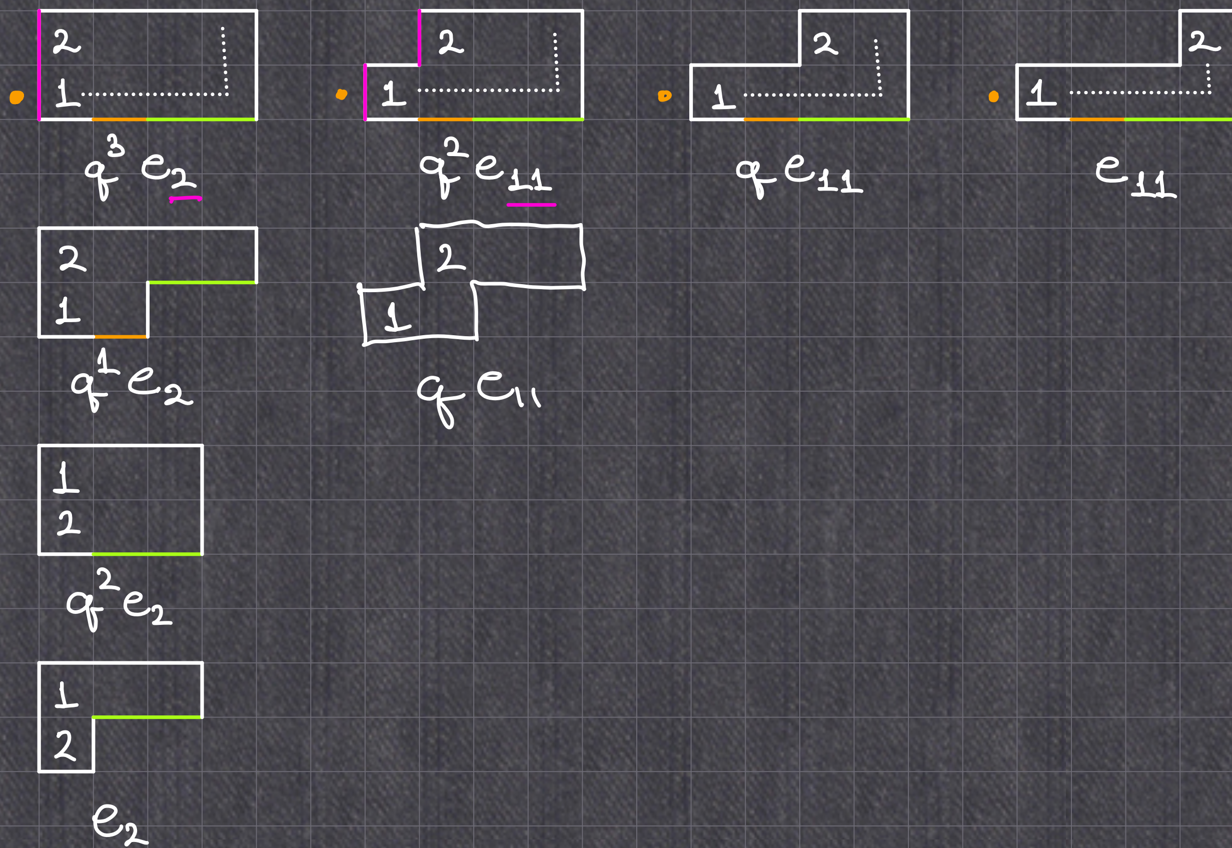
$\Longrightarrow$

Next time, we will

Go to γ-parking functions.

An example: (Finished in talk #3)

Computing $\Delta_{m_2} \boxminus e_{11} \big|_{t=1}$



$\rightarrow \Delta_{m_2} \boxminus e_{11} =$

Now for $\Delta_{m_\gamma} \boxminus s_\lambda$ we look at lattice words of type λ'.

$\Rightarrow \Delta_{m_2} \boxminus s_2$ requires words for the shape $(1,1)$

$\Rightarrow$ the word is $(1,2)$

$\Rightarrow \Delta_{m_2} \boxminus s_2 \big|_{t=1} =$