Delta and Theta operator expansions #3

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Last time we saw that

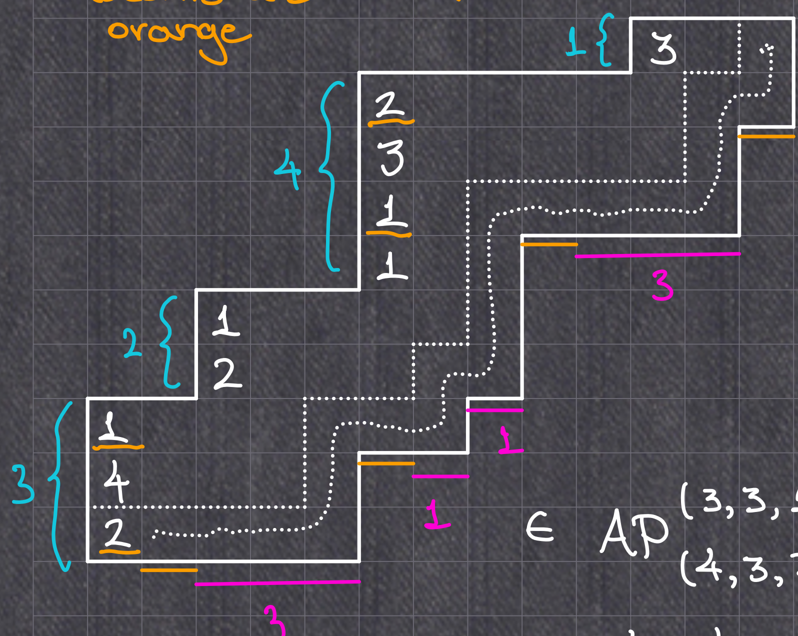
$$\Delta_{m_\gamma} M\Delta_{e_1}\Pi\, e^*_\lambda = \Delta_{m_\gamma} \Xi\, e_\lambda$$

$$= \sum_{P \in AP_\lambda^\gamma} q^{\text{area}(P)} e_{\eta(P)}$$

(ascent compatible labelings of parallelogram polyominoes)

Example.

ascents are labeled orange



λ = multiplicities of labels $= (4,3,2,1)$

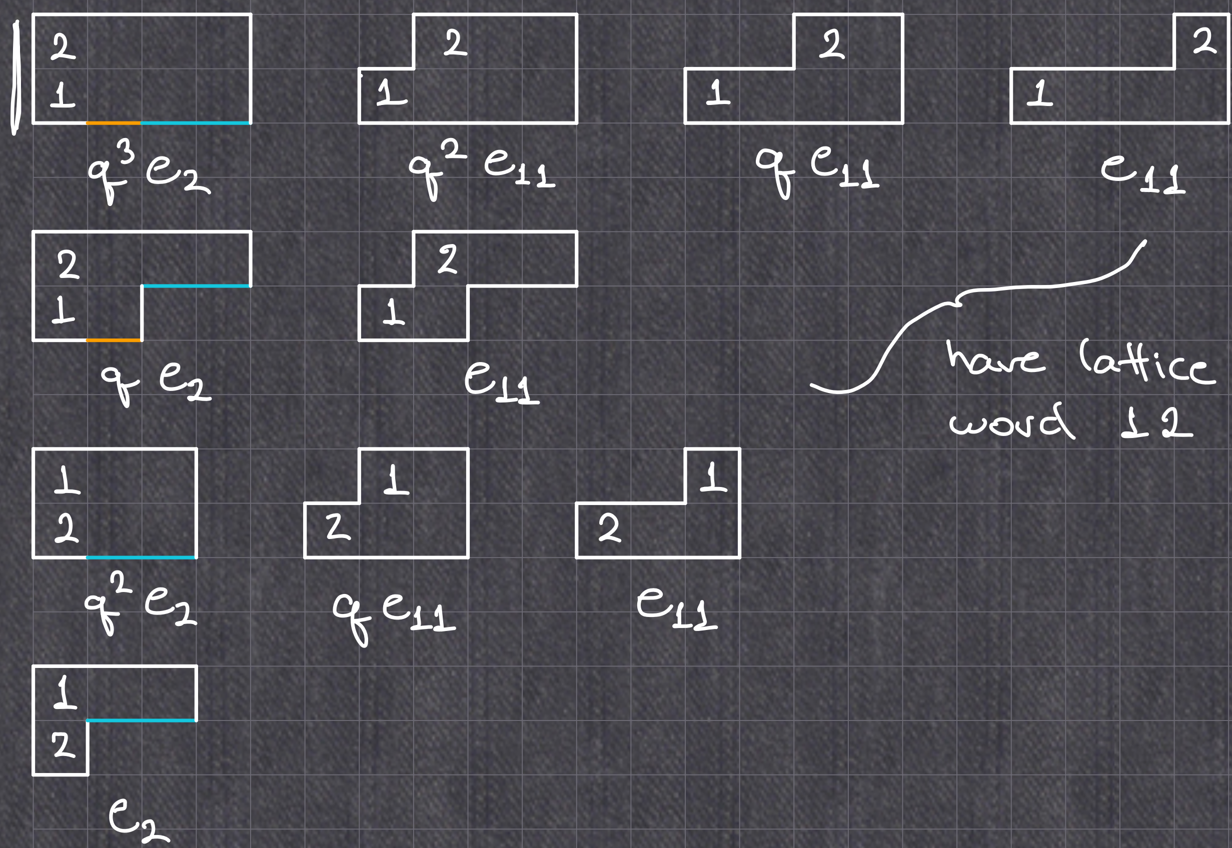
γ = horizontal segments of bottom path $= (3,3,1,1)$

$\in AP^{(3,3,1,1)}_{(4,3,2,1)}$

area $= 4+4+4+5+2+2+6+6+2$

$\eta(P) = (3,2,4,1)$

Example: $\Delta_{m_2} \boxminus e_{11} \big|_{t=1}$



$$\Delta_{m_2} M \Delta_{e_1} \Pi e_{11}^* \big|_{t=1} = (q^3+q^2+q+1)\, e_2 + (q^4+2q^3+2q^2+q)\, e_{11}$$

$$\Delta_{m_2} M \Delta_{e_1} \Pi s_2^* \big|_{t=1} = (q^3+q)\, e_2 + (q^2+q+2)\, e_{11}$$

lattice words for the shape (2) recording columns

entries: 1 2
columns: 1 2

$\boxed{1\ 2}$ Lattice word: 1 2

Special cases. $\Xi e_n = e_n$

$\lambda = n \Rightarrow M \Delta_{e_1} \Pi e_n^* = e_n$

$\Delta_{m_\gamma} \Xi e_n = \Delta_{m_\gamma} e_n$

$\lambda = n \Rightarrow$ labels are all 1's and no ascents.

subcase : $\gamma = (1^n) \Rightarrow \Delta_{m_\gamma} e_n = \nabla e_n$

bottom path is a staircase

$$\nabla e_n \Big|_{t=1} = \sum_{D \in D_n} q^{\text{area}(D)} e_{\eta(D)}$$

$\in D_n$ Dyck paths

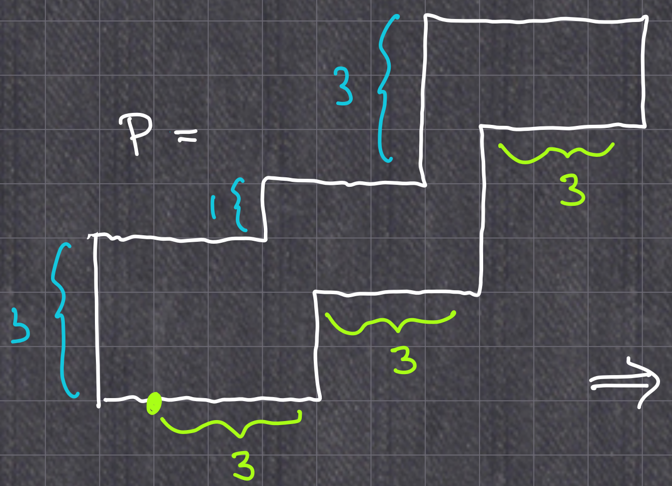
$\eta(D) = (4,2,2)$

$\lambda = n$, γ general

$$\Delta_{m_\gamma} e_n \Big|_{t=1} = \sum_{\text{polyominoes}} q^{\text{area}(P)} e_{\eta(P)}$$

with horizontal segments of bottom path rearranging to γ.

$P =$

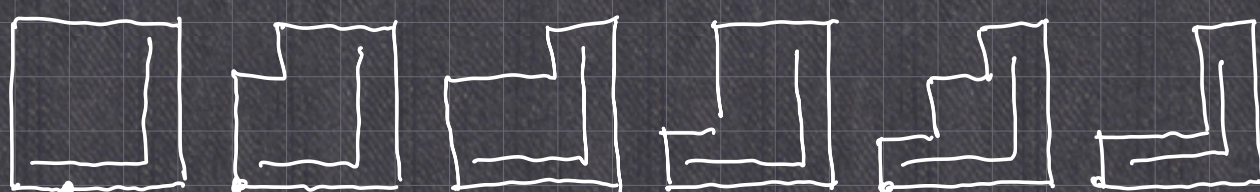


$\eta(P) = (3,1,3)$

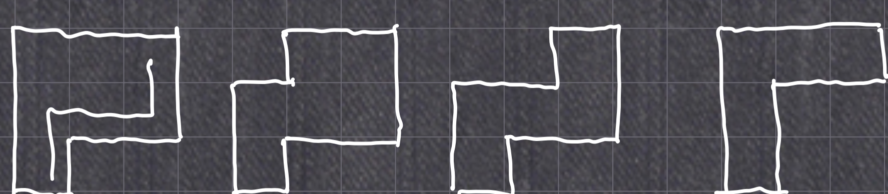
$\gamma = (3,3,3)$

$\Rightarrow$ $\Delta_{m_\gamma} e_\mu \Big|_{t=1} = \cdots$

example: $\gamma = 2$, $n=3$



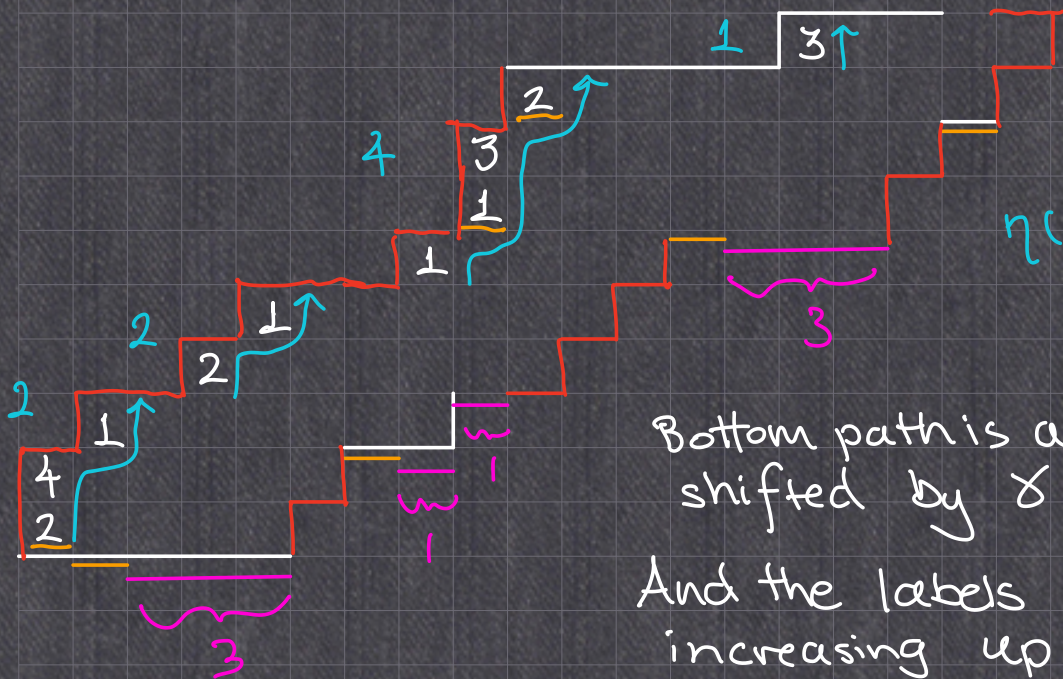
$q^4 e_3$ $\quad$ $q^3 e_{21}$ $\quad$ $q^2 e_{21}$ $\quad$ $q^2 e_{21}$ $\quad$ $q e_{111}$ $\quad$ e_{21}



$q^2 e_3$ $\quad$ $q e_{21}$ $\quad$ e_{21} $\quad$ e_3

$\Delta_{m_2} e_3 =$ the sum.

γ-Parking Functions



$\eta(P) = (2,2,4,1)$

Bottom path is a staircase shifted by γ.

And the labels are increasing up columns

$$\Delta_{m_\gamma} M \Delta_{e_1} \Pi e^*_\lambda \Big|_{t=1} = \sum_{P \in PF^\gamma_\lambda} q^{\text{area}(P)} e_{\eta(P)}$$

Special case: $\lambda = 1^n \Rightarrow$ labels are $1,2,3,\ldots,n$

$\gamma = \emptyset \qquad \Rightarrow \quad P \in P^{1^n}_\lambda$ is a parking function

$\eta(P) = (4,2,2)$

$$M \Delta_{e_1} \Pi\, e^*_{1^n} = \sum_{P \in PF_n} q^{\text{area}(P)} e_{\eta(P)}$$

related to representations of certain quivers. And rooted tiered trees

$$\nabla e_n = \sum_{D \in D_n} q^{\mathrm{area}} e_{\eta(D)}$$

$$M \Delta_{e_1} \Pi e^*_{1^n} = \sum_{P \in PF_n} q^{\mathrm{area}} e_{\eta(D)}$$

How do we get other special cases?

The extended Delta Conjecture

$$\Delta_{e_3 h_3} e_7 \big|_{t=1} \overset{?}{=}$$

The monomial expansion of $e_3 h_4$.

3 green E steps (for e_3)
all in separate rows

3 blue E steps (for h_3)
in any way possible

• 0 valleys

$$\Delta_{e_{n-k} h_m} e_n \big|_{t=1} = \sum q^{\text{area} - *\text{area}} e_{\alpha(P)}$$

* k decorated double N steps

• m valleys

$q^{13} e_{421}$

The e-positivity phenomenon

If F is monomial positive and G is Schur positive, then

$$\Delta_F \,\Xi\, G \Big|_{q \to 1+u}$$

is e-positive (the coefficients are in $\mathbb{N}[u,t]$)

Theorem for $\gamma = (k)$ and $G = e_n$

$$\Delta_{e_k} e_n$$

Conjecture is saying

$$\Delta_{m_\gamma} M \Delta_{e_1} \Pi\, s_\lambda^*$$

exhibits the e-positivity phenomenon.

If this is true, then from what we have seen there should exist a statistic stat so that

$$\Delta_{m_\gamma} M \Delta_{e_1} \Pi e_\lambda^* \Big|_{q \to 1+u} = \sum_{P \in PF_\lambda^\gamma} \sum_{S \subseteq \mathrm{Area}(P)} u^{|S|} t^{\mathrm{stat}(S,P)} e_{\eta(P)}$$