Properties of the nonsymmetric Robinson-Schensted-Knuth algorithm

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Abstract

We introduce a generalization of the Robinson-Schensted-Knuth algorithm to composition tableaux involving an arbitrary permutation. If the permutation is the identity our construction reduces to Mason's original composition Robinson-Schensted-Knuth algorithm. In particular we develop an analogue of Schensted insertion in our more general setting, and use this to obtain new decompositions of the Schur function into nonsymmetric elements (which become Demazure atoms when the permutation is the identity). Other applications include Pieri rules for multiplying these generalized Demazure atoms by complete homogeneous symmetric functions or elementary symmetric functions, a generalization of Knuth's correspondence between matrices of nonnegative integers and pairs of tableaux, and a version of evacuation for arbitrary permutations.

1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{P}$ denote the set of positive integers $\{1, 2, \ldots, \}$. We say that $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a weak composition of m into n parts if each $\gamma_i \in \mathbb{N}$ and $\sum_{i=1}^{n} \gamma_i = m$. Letting $|\gamma| = \sum_i \gamma_i$, the *(column) diagram of* γ is the figure $dg'(\gamma)$ consisting of $|\gamma|$ *cells* arranged into columns so that the i^{th} column contains γ_i cells. For example, the diagram of $\gamma = (2, 0, 1, 0, 3)$ is pictured in Figure 1. The *augmented diagram of* γ, denoted by $\widehat{dg}(\gamma)$, consists of the diagram of γ together with an extra row of n cells attached below. These extra cells are referred to as the *basement* of the augmented diagram. We let $\boldsymbol{\lambda}(\gamma)$ be the partition that results by taking the weakly decreasing rearrangement of the parts of γ. Thus if $\gamma = (2, 0, 1, 0, 3)$, then $\boldsymbol{\lambda}(\gamma) = (3, 2, 1, 0, 0)$.

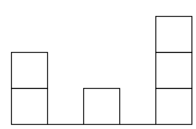


Figure 1: The diagram of $\gamma = (2, 0, 1, 0, 3)$.

Macdonald [6] defined a famous family of symmetric polynomials $P_\lambda(x_1, x_2, \ldots, x_n; q, t)$, which have important applications to a variety of areas. In [5] he showed that many of the properties of the P_λ, such as satisfying a multivariate orthogonality condition, are shared by a family of non-symmetric polynomials $E_\gamma(x_1, \ldots, x_n; q, t)$, where γ is a weak composition with n parts. Haglund, Haiman and Loehr [1] obtained a combinatorial formula for $E_\gamma(x_1, \ldots, x_n; q, t)$ in terms of fillings of $\widehat{dg}(\gamma)$ by positive integers satisfying certain constraints. It will be simpler for us to phrase things in terms of a transformed version of the E_γ studied by Marshall [7] which we denote by $\widehat{E}_\gamma(x_1, \ldots, x_n; q, t)$. The $\widehat{E}_\gamma$ can be obtained from the E_γ by sending $q \to 1/q$, $t \to 1/t$, reversing the x-variables, and reversing the parts of γ. The corresponding combinatorial expression for $\widehat{E}_\gamma(x_1,\ldots,x_n;0;0)$ from [1] involves what Mason [8], [9] later calls semi-skyline augmented fillings. It was previously shown [9] that $\widehat{E}_\gamma(x_1,\ldots,x_n;0,0)$ equals the "standard bases" of Lascoux and Schützenberger [4], which were renamed Demazure atoms to avoid confusion with other objects called "standard bases". Mason introduced a generalization of the RSK algorithm involving semi-skyline augmented fillings, and used this to give combinatorial proofs of several results involving Demazure atoms. For example, her generalized RSK gives a bijective proof that for any partition β,

$$s_\beta(x_1,\ldots,x_n) = \sum_{\substack{\gamma \\ \lambda(\gamma)=\beta}} \widehat{E}_\gamma(x_1,\ldots,x_n;0,0). \tag{1}$$

Mason's extended RSK is also instrumental in work of Haglund, Luoto, Mason and van Willigenburg who developed the theory of a new basis for the ring of quasisymmetric functions called quasisymmetric Schur functions [2], [3]. In particular these authors use it to prove a generalization of the Littlewood-Richardson rule, where the product of a Demazure atom, or Demazure character, or quasisymmetric Schur function and a Schur function is expanded in terms of Demazure atoms, Demazure characters, or quasisymmetric Schur functions, respectively.

Let ϵ_n denote the identity $1\ 2\ \cdots\ n$ in S_n and $\bar{\epsilon}_n$ the reverse of the identity $n\ n-1\ \cdots\ 1$. In [8], [9] and in [2],[3] the basements of the diagrams $\widehat{dg}(\gamma)$ are always filled by either ϵ_n (i.e., i is in the ith column of the basement), or by $\bar{\epsilon}_n$. In this article we show that many of the nice properties (such as an insertion algorithm) of Mason's extended RSK hold with the basement consisting of an arbitrary permutation $\sigma \in S_n$. In particular we define a weight preserving bijection which shows

$$s_\beta(x_1,\ldots,x_n) = \sum_\gamma \widehat{E}^\sigma_\gamma(x_1,\ldots,x_n) \tag{2}$$

where the sum is over all weak compositions γ such that $\boldsymbol{\lambda}(\gamma) = \beta$ and $\gamma_i \geq \gamma_j$ whenever $i < j$ and $\sigma_i > \sigma_j$. Here $\widehat{E}^\sigma_\gamma(x_1,\ldots,x_n)$ is the version of $\widehat{E}_\gamma(x_1,\ldots,x_n;0,0)$ with basement σ. In the special case when $\sigma = \bar{\epsilon}_n$ there is only one term in the sum above so that $E^{\bar{\epsilon}_n}_\beta = s_\beta$, while if σ is ϵ_n then (2) reduces to (1).

As with Mason's insertion algorithm, we shall see that our insertion algorithm with general basement also commutes in a natural way with the RSK algorithm. This useful fact will allow us extend results of Mason to our more general setup. Moreover, we shall give a precise characterization of how the results of our insertion algorithm vary as the basement σ varies. If $\sigma = \bar{\epsilon}_n$ our algorithm becomes essentially equivalent to ordinary RSK insertion, while if $\sigma = \epsilon_n$, it reduces to Mason's insertion algorithm.

The outline of this paper is as follows. In section 2, we formally define the objects we will be working with, namely permuted basement semi-standard augmented fillings relative to a permutation σ (PBF's). In sections 3 and 4, we describe our insertion algorithm for PBF's and derive its general properties. In section 5, we use it to prove analogues of the Pieri rules for the product of a homogeneous symmetric function $h_n(x_1,\ldots,x_n)$ times an $\widehat{E}^\sigma_\gamma(x_1,\ldots,x_n)$ and the product of a elementary symmetric function $e_n(x_1,\ldots,x_n)$ times an $\widehat{E}^\sigma_\gamma(x_1,\ldots,x_n)$. In section 6 we develop an analogue of the RSK algorithm for permuted basements and prove several of its basic properties. Finally, in section 7 we study the analogue of evacuation for PBF's.

2 Permuted basement semi-standard augmented fillings.

The positive integer n is fixed throughout, while γ will always denote a weak composition into n parts and σ a permutation in S_n. We let (i,j) denote the cell in the i-th column, reading from left to right, and the j-th row, reading from bottom to top, of $\widehat{dg}(\gamma)$. The basement cells of $\widehat{dg}(\gamma)$ are considered to be in row 0 so that $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(i,0) : 1 \leq i \leq n\}$. The *reading order* of the cells of $\widehat{dg}(\gamma)$ is obtained by reading the cells in rows from left to right, beginning at the highest row and reading from top to bottom. Thus a cell $a = (i,j)$ is less than a cell $b = (i',j')$ in the reading order if either $j > j'$ or $j = j'$ and $i < i'$. For example, if $\gamma = (0,2,0,3,1,2,0,0,1)$, then $\widehat{dg}(\gamma)$ is pictured in Figure 2 where we have placed the number i in the i-th cell in reading order. An *augmented filling, F*, of an augmented diagram $\widehat{dg}(\gamma)$ is a function $F : \widehat{dg}(\gamma) \to \mathbb{P}$, which we picture as an assignment of positive integers to the cells of $\widehat{dg}(\gamma)$. We let $F(i,j)$ denote the entry in cell (i, j) of F. The reading word of F, *read*(F), is obtained by recording the entries of F in the reading order of $dg'(\gamma)$. The *content* of F is the multiset of entries which appear in the filling F. Throughout this article, we will only be interested in fillings F such that entries in each column are weakly increasing reading from top to bottom and the basement entries form a permutation in the symmetric group S_n.

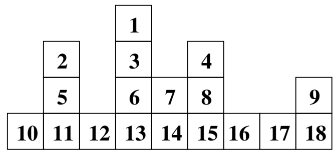


Figure 2: The reading word order of the cells of the augmented filling for $\gamma = (0, 2, 0, 3, 1, 2, 0, 0, 1)$.

Next we define type A and B triples as in [8]. A *type A triple* in an augmented diagram of shape γ is a set of three cells a, b, c of the form $(i, k), (j, k), (i, k-1)$ for some pair of columns $i < j$ of the diagram and some row $k > 0$, where $\gamma_i \geq \gamma_j$. A *type B triple* is a set of three cells a, b, c of the form $(j, k+1), (i, k), (j, k)$ for some pair of columns $i < j$ of the diagram and some row $k \geq 0$, where $\gamma_i < \gamma_j$. Note that basement cells can be elements of triples. As noted above, in this article our fillings F have weakly increasing row entries top-to-bottom, so we always have the entry values satisfying $F(a) \leq F(c)$. We say that a triple of either type is an *inversion* triple if the relative order of the entries is either $F(b) < F(a) \leq F(c)$ or $F(a) \leq F(c) < F(b)$. Otherwise we say that the triple is a *coinversion* triple, i.e. if $F(a) \leq F(b) \leq F(c)$. Figure 3 depicts type A and B triples.

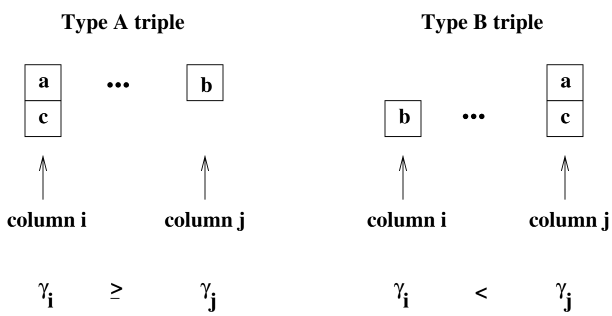


Figure 3: Type A and B triples.

A *semi-standard augmented filling* is a filling of an augmented diagram with positive integer entries so that (i) the column entries are weakly increasing from top to bottom, (ii) the basement entries form a permutation of $1, 2, \ldots, n$ if there are n cells in the basement, and (iii) every Type A or B triple is an inversion triple. We say that cells $c_1 = (x_1, y_1)$ and $c_2 = (x_2, y_2)$ are *attacking* if either c_1 and c_2 lie in the same row, i.e. $y_1 = y_2$, or if c_1 lies strictly to the left and one row below c_2, i.e. if $x_1 < x_2$ and $y_2 = y_1 + 1$. We say that filling F is *non-attacking* if $F(c_1) \neq F(c_2)$ whenever c_1 and c_2 are attacking. It is easy to see from our definition of inversion triples that a semi-standard augmented filling F must be non-attacking. A superscript σ on a filling F, as in F^σ, means the basement entries form the permutation σ.

We say that a filling F^σ is a *permuted basement semi-standard augmented filling* (PBF) of shape γ with basement permutation σ if
(I) F^σ is a semi-standard augmented filling of $\widehat{dg}(\gamma)$,
(II) $F^\sigma((i, 0)) = \sigma_i$ for $i = 1, \ldots, n$, and
(III) for all cells $a = F^\sigma(i_2, j), b = F^\sigma(i_1, j-1)$ such that $i_1 < i_2$ and $\gamma_{i_1} < \gamma_{i_2}$, we have $b < a$.
We shall call condition (III) the *B-increasing* condition, as pictured in Figure 4.

Given a PBF F^σ of shape γ , we define the weight of F^σ, $W(F^\sigma)$, to be

$$W(F^\sigma) = \prod_{(i,j) \in dg'(\gamma)} x_{F^\sigma(i,j)}. \tag{3}$$

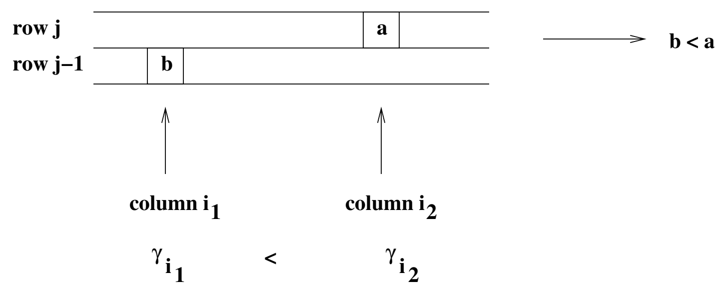


Figure 4: The B-increasing condition.

Let $\mathcal{PBF}(\gamma,\sigma)$ denote the set of all PBFs F^σ of shape γ with basement σ. We then define

$$\widehat{E}_\gamma^\sigma(x_1,x_2,\ldots,x_n) = \sum_{F^\sigma\in\mathcal{PBF}(\gamma,\sigma)} W(F^\sigma). \tag{4}$$

The following fact about PBFs will be used frequently in the sequel.

Lemma 1. *Let F^σ be a* PBF *of shape γ and assume that $i<m$ and $\gamma_i \geq \gamma_m$. If $F^\sigma(i,j) > F^\sigma(m,j)$ for some $j\geq 0$, then $F^\sigma(i,k) > F^\sigma(m,k)$ for all $k\geq j$.*

Proof. Arguing for a contradiction, suppose that the Lemma fails. Then let k be the smallest ℓ such that $\ell \geq j$ and $F^\sigma(i,\ell) \leq F^\sigma(m,\ell)$. This implies the triple $\{(i,k),(m,k),(i,k-1)\}$ is a type A coinversion triple since

$$F^\sigma(i,k) \leq F^\sigma(m,k) \leq F^\sigma(m,k-1) < F^\sigma(i,k-1).$$

Since we are assuming that F^σ has no type A coinversion triples, there can be no such k. □

We end this section by considering the two special cases of PBFs where the basement is either the identity or the reverse of the identity. In the special case where the basement permutation $\sigma=\epsilon_n$, a PBF is a semi-standard augmented filling as defined by Mason in [8]. Next consider the case where $F^{\bar{\epsilon}_n}$ is a PBF of shape γ with basement $\bar{\epsilon}_n$. We claim this implies $\gamma_i \geq \gamma_{i+1}$ for all i. For suppose $\gamma_i < \gamma_{i+1}$. Then $(i+1,1)$ must be a cell in $F^{\bar{\epsilon}_n}$ and $F^{\bar{\epsilon}_n}(i+1,1) \leq F^{\bar{\epsilon}_n}(i+1,0) = n-i < n-i+1 = F^{\bar{\epsilon}_n}(i,0)$ so that the cells $b=(i,0)$ and $a=(i+1,1)$ would violate the B-increasing condition for $F^{\bar{\epsilon}_n}$. Thus in the special case of a PBF $F^{\bar{\epsilon}_n}$, the shape γ must be a partition, i.e. we must have $\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_n$. But then Lemma 1 implies that the entries of $F^{\bar{\epsilon}_n}$ must strictly decrease in rows reading from left to right. Since the elements of $F^{\bar{\epsilon}_n}$ must weakly decrease in columns reading from bottom to top, we see that $F^{\bar{\epsilon}_n}$ is what could be called a reverse row strict tableau with basement $\bar{\epsilon}_n$ attached. It follows that for γ a partition, $\widehat{E}_\gamma^{\bar{\epsilon}_n}(x_1,x_2,\ldots,x_n)$ is equal to the Schur function $s_\gamma(x_1,x_2,\ldots,x_n)$.

3 An analogue of Schensted insertion

In [8], Mason defined a procedure $k\to F$ to insert a positive integer k into a semi-skyline augmented filling, which is a PBF with basement permutation equal to the identity. In this section, we shall describe an extension of this insertion procedure which inserts a positive integer into a PBF with an arbitrary basement permutation.

Let F^σ be a PBF with basement permutation $\sigma\in S_n$. We shall define a procedure $k\to F^\sigma$ to insert a positive integer k into F^σ. Let $\bar{F}^\sigma$ be the extension of F^σ which first extends the basement permutation σ by adding j in cell $(j,0)$ for $n<j\leq k$ and then adds a cell which contains a 0 on top of each column. Let $(x_1,y_1),(x_2,y_2),\ldots$ be the cells of this extended diagram listed in reading order. Formally, we shall define the insertion procedure of $k\to(x_1,y_1),(x_2,y_2),\ldots$ of k into the sequence of cells $(x_1,y_1),(x_2,y_2),\ldots$.

Let $k_0=k$ and look for the first i such that $\bar{F}^\sigma(x_i,y_i) < k_0 \leq \bar{F}^\sigma(x_i,y_i-1)$. Then there are two cases.

Case 1. If $\bar{F}^\sigma(x_i, y_i) = 0$, then place k_0 in cell (x_i, y_i) and terminate the procedure.

Case 2. If $\bar{F}^\sigma(x_i, y_i) \neq 0$, then place k_0 in cell (x_i, y_i), set $k_0 := \bar{F}^\sigma(x_i, y_i)$ and repeat the procedure by inserting k_0 into the the sequence of cells $(x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), \ldots$. In such a situation, we say that $\bar{F}^\sigma(x_i, y_i)$ is *bumped* in the insertion $k \to F^\sigma$.

The sequence of cells that contain elements which are bumped in the insertion $k \to F^\sigma$ plus the final cell which is added when the procedure is terminated will be called the *bumping path* of the insertion. For example, Figure 5 shows an extended diagram of a PBF with basement permutation equal to 6 1 3 4 2 5. If we insert 5 into this PBF, then it is easy to see that the first element bumped is the 4 in column 1. Therefore this 4 will be replaced by 5 and we will insert 4 into the remaining sequence of cells. The first element that 4 can bump is the 2 in column 4. Thus that 4 will replace the 2 in column 4 and 2 will be inserted in the remaining cells. But then that 2 will bump the 0 in column 5 so that the procedure will terminate. Thus the circled elements in Figure 5 correspond to the bumping path of this insertion. Notice that the entries of $\bar{F}^\sigma$ in the bumping path must strictly decrease as we proceed in reading order.

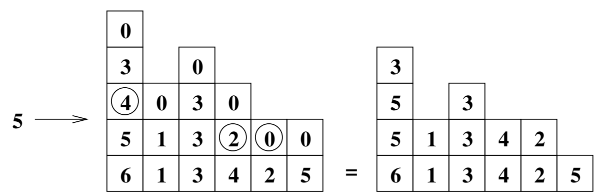


Figure 5: The bumping path of an insertion into a PBF.

The following lemmas are needed in order to prove that the insertion procedure terminates and the result is a PBF.

Lemma 2. *Let $c_1 = (i_1, j_1)$ and $c_2 = (i_2, j_2)$ be two cells in a* PBF F^σ *such that $F^\sigma(c_1) = F^\sigma(c_2) = a$, c_1 appears before c_2 in reading order, and no cell between c_1 and c_2 in reading order contains the entry a. Let $c_1' = (i_1', j_1')$ and $c_2' = (i_2', j_2')$ be the cells in $k \to F^\sigma$ containing the entries from c_1 and c_2 respectively. Then $j_1' > j_2'$.*

Proof. Consider the cell $\underline{c_1} = (i_1, j_1 - 1)$ immediately below c_1 in the diagram F^σ. Note that c_1 attacks all cells of F^σ to its right that lie in same row as well as all cells to its left that lie one row below the row of c_1. Since entries in cells which are attacked by c_1 must be different from $F^\sigma(c_1)$, it follows that c_2 must appear weakly after $\underline{c_1}$ in reading order. If $c_2 = \underline{c_1} = (i_1, j_1 - 1)$, then the entry in cell c_1 cannot be bumped because that would require $F^\sigma(i_1, j_1) < k_0 \leq F^\sigma(i_1, j_1 - 1)$. Thus either c_2 is not bumped in which case the Lemma automatically holds or c_2 is bumped in which case its entry ends up in a cell which is later in reading order so that $j_1 = j_1' > j_1 - 1 \geq j_2'$.

Thus we may assume that $F^\sigma(\underline{c_1}) > F^\sigma(c_1)$ and that c_2 follows $\underline{c_1}$ in reading order. This means that the element $\overline{c_2} = (i_2, j_2 + 1)$ which lies immediately above c_2 follows c_1 in reading order and the entry in cell $\overline{c_2}$ must be strictly less than a by our choice of c_2. If the entry in c_1 is not bumped, then again we can conclude as above that the entry in c_2 will end up in a cell which follows $\underline{c_1}$ in reading order so that again $j_1 = j_1' > j_1 - 1 \geq j_2'$. Finally, suppose that the entry a in cell c_1 is bumped. Since $F^\sigma(\overline{c_2}) < a = F^\sigma(c_2)$, it follows that $F^\sigma(\overline{c_2})$ is a candidate to be bumped by a. Thus the a that was bumped out of cell c_1 must end up in a cell which weakly precedes $\overline{c_2}$ in reading order and hence it ends up in a row which is higher than the row of c_2. But then the entry in c_2 either is not bumped or ends up in a cell which follows c_2 in reading order so that the final resting place of the entry in c_2 lies in a row weakly below the row of c_2. Thus the lemma holds. □

Lemma 3. *Suppose that F^σ is a* PBF *and k is a positive integer. Then every type A triple in $k \to F^\sigma$ is an inversion triple.*

Proof. Suppose that F^σ is of shape $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $n \geq k$. Consider an arbitrary type A triple $\{a = (x_1, y_1), b = (x_2, y_1), c = (x_1, y_1 - 1)\}$ in $\tilde{F}^\sigma := k \to F^\sigma$. If $\{a, b, c\}$ form a coinversion in $\tilde{F}^\sigma$, then

$\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b) \leq \tilde{F}^\sigma(c)$. Since the entries in the bumping path in the insertion $k \to F^\sigma$ form a strictly decreasing sequence, only one of $\{F^\sigma(a), F^\sigma(b), F^\sigma(c)\}$ can be bumped by the insertion procedure $k \to F^\sigma$. Let $\bar{F}^\sigma$ be the extended diagram corresponding to F^σ as defined in our definition of the insertion $k \to F^\sigma$. We claim that the triple conditions for F^σ imply that either $\bar{F}^\sigma(b) < \bar{F}^\sigma(a) \leq \bar{F}^\sigma(c)$ or $\bar{F}^\sigma(a) \leq \bar{F}^\sigma(c) < \bar{F}^\sigma(b)$. This follows from the fact that F^σ is a PBF if a, b, c are cells in F^σ. Since the shape of $\tilde{F}^\sigma$ arises from γ by adding a single cell on the outside of γ, we know that c is a cell in F^σ. However, it is possible that exactly one of a or b in not in F^σ and is filled with a 0 in $\bar{F}^\sigma$. If it is b, then we automatically have $\bar{F}^\sigma(b) < \bar{F}^\sigma(a) \leq \bar{F}^\sigma(c)$. If it is a, then the column that contains a is strictly shorter than the column that contains b because in $\tilde{F}^\sigma$, it must the be case that the height of column x_1 is greater than or equal to the height of column x_2 since $\{a, b, c\}$ is a type A triple in $\tilde{F}^\sigma$. But then the B-increasing condition for F^σ forces $\bar{F}^\sigma(c) < \bar{F}^\sigma(b)$ and, hence, $\bar{F}^\sigma(a) \leq \bar{F}^\sigma(c) < \bar{F}^\sigma(b)$ must hold.

We now consider two cases.

Case 1. $\bar{F}^\sigma(b) < \bar{F}^\sigma(a) \leq \bar{F}^\sigma(c)$.

Note in this case, $0 < \bar{F}^\sigma(a)$ so that a is a cell in F^σ. Moreover the entries in a and c cannot be bumped in the insertion $k \to F^\sigma$ since their replacement by a larger value would not produce the desired ordering $\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b) \leq \tilde{F}^\sigma(c)$. Thus it must be the case that $\tilde{F}^\sigma(b)$ was bumped in the insertion $k \to F^\sigma$. We now consider two subcases.

Subcase 1.a. $\bar{F}^\sigma(a) < \tilde{F}^\sigma(b)$.

We know that $\tilde{F}^\sigma(b)$ bumps $\bar{F}^\sigma(b)$. We wish to determine where $\tilde{F}^\sigma(b)$ came from in the insertion process $k \to F^\sigma$. It cannot be that $\tilde{F}^\sigma(b) = k$ or that it was bumped from a cell that comes before a in the reading order since it would then meet the conditions to bump the entry $F^\sigma(a)$ in cell a as $F^\sigma(a) < \tilde{F}^\sigma(b) \leq F^\sigma(c)$. Thus it must have been bumped from a cell after a but before b in reading order. That is, $\tilde{F}^\sigma(b) = F^\sigma(d)$ where $d = (x_3, y_1)$ and $x_1 < x_3 < x_2$. Thus we have the situation pictured in Figure 6.

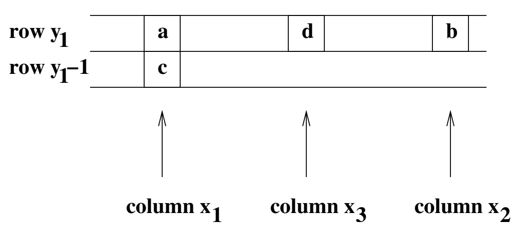


Figure 6: Picture for Subcase 1.a.

However, this is not possible since if $\gamma_{x_1} \geq \gamma_{x_3}$, then the entries in cells a, d, and c would violate the A-triple condition for F^σ and, if $\gamma_{x_1} < \gamma_{x_3}$, then the entries in cells c and d would violate the B-increasing condition on F^σ.

Subcase 1.b $\bar{F}^\sigma(a) = \tilde{F}^\sigma(b)$.

Again we must determine where $\tilde{F}^\sigma(b)$ came from in the insertion process $k \to F^\sigma$. To this end, let r be the least row such that $r > y_1$ and $\bar{F}^\sigma(x_1, r) < \bar{F}^\sigma(x_1, r-1)$. Then we will have the situation pictured in Figure 7 where d is the cell in column x_1 and row r. Thus all the entries of F^σ in the cells in column x_1 between a and d are equal to $F^\sigma(a)$.

Now the region of shaded cells pictured in Figure 7 are cells which are attacked or attack some cell which is equal to $F^\sigma(a)$ and hence their entries in F^σ must all be different from $F^\sigma(a)$. Thus $\tilde{F}^\sigma(b)$ cannot have come from any of these cells since we are assuming that $\bar{F}^\sigma(a) = \tilde{F}^\sigma(b)$. Thus $\tilde{F}^\sigma(b)$ must have come from a cell before d in reading order. But this is also impossible because $\tilde{F}^\sigma(b)$ would then meet the conditions to bump $\bar{F}^\sigma(d)$ which would violate our assumption that it bumps $F^\sigma(b)$.

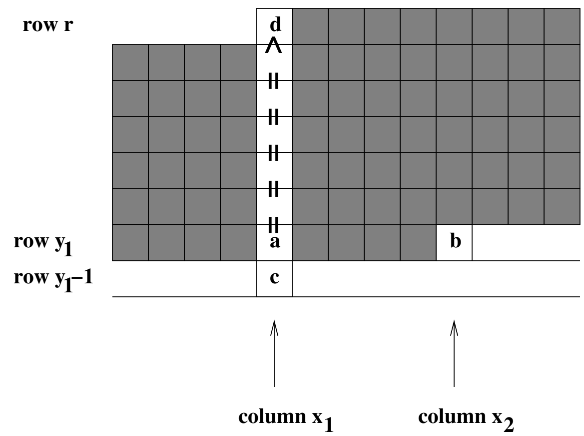


Figure 7: Picture for Subcase 1.b.

Case 2. $\bar{F}^\sigma(a) \leq \bar{F}^\sigma(c) < \bar{F}^\sigma(b)$.

The entry in cell c is the only entry which could be bumped in the insertion $k \to F^\sigma$ if we are to end up with the relative ordering $\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b) \leq \tilde{F}^\sigma(c)$. Since $F^\sigma(c)$ is bumped, this means that c is not in the basement. But if we do not bump either a or b in the insertion $k \to F^\sigma$ and a and b are cells in $\tilde{F}^\sigma$, it must be the case that a and b are cells in F^σ and that there is no change in the heights of columns x_1 and x_2. Thus $\gamma_{x_1} \geq \gamma_{x_2}$. Let $\underline{c}$ be the cell immediately below c and $\underline{b}$ be the cell immediately below b. Thus we must have $F^\sigma(c) < \tilde{F}^\sigma(c) \leq F^\sigma(\underline{c})$. We now consider two subcases.

Subcase 2.a. $\tilde{F}^\sigma(c) = F^\sigma(b)$.

Let r be the least row such that $r > y_1$ and $\bar{F}^\sigma(x_2, r) < \bar{F}^\sigma(x_2, r-1)$. Then we will have the situation pictured in Figure 8 where d is the cell in column x_2 and row r. Thus all the entries of F^σ in the cells in column x_2 between b and d are equal to $F^\sigma(b)$.

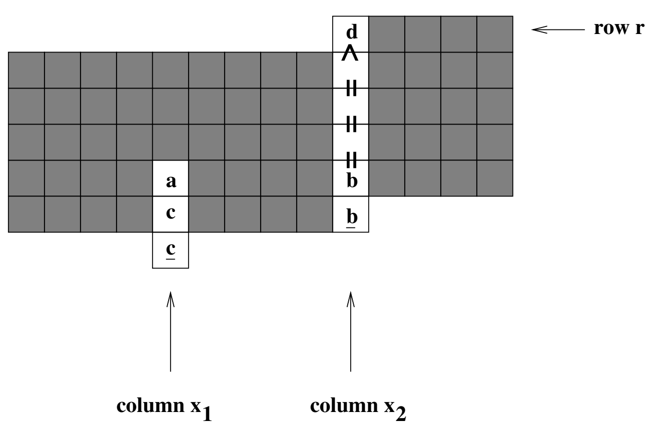


Figure 8: Picture for Subcase 2.a.

Now the region of shaded cells pictured in Figure 8 are cells which are attacked or attack some cell which is equal to $F^\sigma(b)$ and hence their entries in F^σ must all be different from $F^\sigma(b)$. Thus $\tilde{F}^\sigma(c)$ cannot have come from any of these cells since we are assuming that $F^\sigma(b) = \tilde{F}^\sigma(c)$. Hence $\tilde{F}^\sigma(c)$ must have come from a cell before d in reading order. But this is also impossible because $\tilde{F}^\sigma(c)$ would then meet the conditions to bump $\bar{F}^\sigma(d)$ which would violate our assumption that it bumps $F^\sigma(c)$.

Subcase 2.b. $F^\sigma(b) < \tilde{\mathrm{F}}^\sigma(c)$.

First consider the A-triple $c, \underline{c}, \underline{b}$. We cannot have that $F^\sigma(\underline{b}) < F^\sigma(c) \leq F^\sigma(\underline{c})$ since that would imply $F^\sigma(b) \leq F^\sigma(\underline{b}) < F^\sigma(c)$, which would violate our assumption that $F^\sigma(a) < F^\sigma(c) < F^\sigma(b)$. Thus it must be the case that $F^\sigma(c) \leq F^\sigma(\underline{c}) < F^\sigma(\underline{b})$. But then we would have $F^\sigma(b) < \tilde{\mathrm{F}}^\sigma(c) \leq F^\sigma(\underline{c}) < F^\sigma(\underline{b})$ which would mean that $\tilde{\mathrm{F}}^\sigma(c)$ satisfies the conditions to bump $F^\sigma(b)$. Since it does not bump $F^\sigma(b)$, it must be the case that $\tilde{\mathrm{F}}^\sigma(c)$ came from a cell which is after b in the reading order. We now consider two more subcases.

Subcase 2.b.i. $\tilde{\mathrm{F}}^\sigma(c)$ is in the same row as $F^\sigma(b)$.

Assume that $\tilde{\mathrm{F}}^\sigma(c) = F^\sigma(d)$ where $d = (x_3, y_1)$ and $x_2 < x_3$. It cannot be that $\gamma_{x_2} < \gamma_{x_3}$ since then the B-increasing condition would force that $F^\sigma(\underline{b}) < F^\sigma(d) = \tilde{\mathrm{F}}^\sigma(c)$. But that would mean that $F^\sigma(\underline{b}) < \tilde{\mathrm{F}}^\sigma(c) \leq F^\sigma(\underline{c})$ which violates the fact that $F^\sigma(c) \leq F^\sigma(\underline{c}) < F^\sigma(\underline{b})$. Thus it must be the case that $\gamma_{x_2} \geq \gamma_{x_3}$ and, hence, $b, \underline{b}, d$ is a type A triple. As we cannot have $F^\sigma(\underline{b}) < F^\sigma(d) = \tilde{\mathrm{F}}^\sigma(c)$, it must be that case that $\tilde{\mathrm{F}}^\sigma(c) = F^\sigma(d) < F^\sigma(b) \leq F^\sigma(\underline{b})$. But this is also impossible because we are assuming that $F^\sigma(b) < \tilde{\mathrm{F}}^\sigma(c)$.

Subcase 2.b.ii. $\tilde{\mathrm{F}}^\sigma(c)$ is in the same row as $F^\sigma(c)$.

In this case, let $e_1, \ldots, e_s, e_{s+1} = c$ be the cells in the bumping path of the insertion of $k \to F^\sigma$ in row $y_1 - 1$, reading from left to right. Thus we are assuming that $\tilde{\mathrm{F}}^\sigma(c) = F^\sigma(e_s)$. For each e_i, we let $\underline{e}_i$ be the cell directly below e_i and $\overline{e}_i$ be the cell directly above e_i. Thus we have the picture in Figure 9 where we are assuming that $s = 3$ and we have circled the elements in the bumping path.

$\bar{e}_1$			$\bar{e}_2$			$\bar{e}_3$		a				b	
e_1			e_2			e_3		c				$\underline{b}$	
$\underline{e}_1$			$\underline{e}_2$			$\underline{e}_3$		$\underline{c}$					

Figure 9: Picture for Subcase 2.b.ii.

Since the elements in the bumping path decrease, we have that $F^\sigma(e_1) > \cdots > F^\sigma(e_s) > F^\sigma(e_{s+1}) = F^\sigma(c)$ and that for each i, $F^\sigma(e_{i+1}) < F^\sigma(e_i) \leq F^\sigma(\underline{e}_{i+1})$. Let $e_j = (z_j, y_1 - 1)$. Thus $z_{s+1} = x_1$. We claim that we cannot have that $\gamma_{z_i} < \gamma_{z_{i+1}}$ for any i since the B-increasing condition would force $F^\sigma(e_{i+1}) > F^\sigma(\underline{e}_i) \geq F^\sigma(e_i)$. Thus we must have $\gamma_{z_1} \geq \cdots \geq \gamma_{z_s} \geq \gamma_{x_1} \geq \gamma_{x_2}$. Now consider the triples $\{e_i, \underline{e}_i, \underline{b}\}$. We are assuming that $F^\sigma(c) = F^\sigma(e_{s+1}) \leq F^\sigma(\underline{e}_{s+1}) = F^\sigma(\underline{c}) < F^\sigma(\underline{b})$. But since $F^\sigma(e_{s+1}) < F^\sigma(e_s) \leq F^\sigma(\underline{e}_{s+1})$, the $\{e_s, \underline{e}_s, \underline{b}\}$ A-triple condition forces $F^\sigma(e_s) \leq F^\sigma(\underline{e}_s) < F^\sigma(\underline{b})$. Now if e_{s-1} exists, then we know that $F^\sigma(e_s) < F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_s)$ and, hence, the $\{e_{s-1}, \underline{e}_{s-1}, \underline{b}\}$ A-triple condition also implies that $F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_{s-1}) < F^\sigma(\underline{b})$. If e_{s-2} exists, then we know that $F^\sigma(e_{s-1}) < F^\sigma(e_{s-2}) \leq F^\sigma(\underline{e}_{s-1})$ and, hence, the $\{e_{s-2}, \underline{e}_{s-2}, \underline{b}\}$ A-triple condition must also imply that $F^\sigma(e_{s-2}) \leq F^\sigma(\underline{e}_{s-2}) < F^\sigma(\underline{b})$. Continuing on in this way, we can conclude that for all j, $F^\sigma(e_j) \leq F^\sigma(\underline{e}_j) < F^\sigma(\underline{b})$. Next consider the $\overline{e}_i, e_i, b$ A-triple conditions. We are assuming that $F^\sigma(b) < \tilde{\mathrm{F}}^\sigma(c) = F^\sigma(e_s)$. Thus it must be the case that $F^\sigma(b) < F^\sigma(\overline{e}_s) \leq F^\sigma(e_s)$. But since $F^\sigma(e_s) < F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_s)$, the $\overline{e}_{s-1}, e_{s-1}, b$ A-triple condition implies that $F^\sigma(b) < F^\sigma(\overline{e}_{s-1}) \leq F^\sigma(e_{s-1})$. If e_{s-2} exists, then since $F^\sigma(e_{s-1}) < F^\sigma(e_{s-2}) \leq F^\sigma(\underline{e}_{s-1})$, the $\overline{e}_{s-2}, e_{s-2}, b$ A-triple condition also implies that $F^\sigma(b) < F^\sigma(\overline{e}_{s-2}) \leq F^\sigma(e_{s-2})$. Continuing on in this way, we can conclude that for all j, $F^\sigma(b) < F^\sigma(\overline{e}_j) \leq F^\sigma(e_j)$.

Thus in this case, we must have $F^\sigma(b) < F^\sigma(\overline{e}_1) \leq F^\sigma(e_1) \leq F^\sigma(\underline{e}_1) < F^\sigma(\underline{b})$. Now the question is where can the element z which bumps $F^\sigma(e_1)$ come from? We claim that z cannot equal k or come from a cell before b in reading order since it satisfies the condition to bump b and b is not bumped. Thus it must have come from a cell $d = (x_3, y_1)$ which lies in the same row as b but comes after b in reading order. In that case, we must have $F^\sigma(e_1) < F^\sigma(d) \leq F^\sigma(\underline{e_1}) < F^\sigma(\underline{b})$. Thus it cannot be that $\gamma_{x_2} < \gamma_{x_3}$ since the B-increasing condition would force $F^\sigma(\underline{b}) < F^\sigma(\underline{d})$. Thus $\gamma_{x_2} \geq \gamma_{x_3}$. But in that case, we would have

$F^\sigma(b) < F^\sigma(d) < F^\sigma(\underline{b})$ which would be a coinversion A triple in F^σ.

Thus we have shown that in Subcase 2, c could not have been bumped and, hence, there can be no coinversion A triples in $k \to F^\sigma$. □

Lemma 4. *Every type B triple in $k \to F^\sigma$ is an inversion triple.*

Proof. Suppose that F^σ is of shape $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $n \geq k$. Consider an arbitrary type B triple $\{b = (x_1, y_1), a = (x_2, y_1+1), c = (x_2, y_1)\}$ in $\tilde{F}^\sigma := k \to F^\sigma$. If $\{a, b, c\}$ is a coinversion triple in $\tilde{F}^\sigma$, then $\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b) \leq \tilde{F}^\sigma(c)$.

We claim that it cannot be that $\gamma_{x_1} = \gamma_{x_2}$. That is, we are assuming that in $\tilde{F}^\sigma$, the height of column x_1 is less than the height of column x_2 so that if $\gamma_{x_1} = \gamma_{x_2}$, then we must have added a cell on top of column x_2 in the insertion $k \to F^\sigma$. Now if c is not at the top of its column in F^σ, then a is a cell in F^σ and there must be a cell $\bar{b}$ on top of b in F^σ. Moreover the entries in cells a, b, c, and $\bar{b}$ did not change in the insertion $k \to F^\sigma$. But then $\{b, \bar{b}, a\}$ is a type A triple in F^σ and since $F^\sigma(a) \leq F^\sigma(b)$, the $\{b, \bar{b}, a\}$ A-triple condition implies that $F^\sigma(a) < F^\sigma(\bar{b}) \leq F^\sigma(b)$. But then Lemma 1 would force that $F^\sigma(x_1, \gamma_{x_1}) > F^\sigma(x_2, \gamma_{x_1})$. Moreover, the B-increasing condition would force $F^\sigma(x_1, \gamma_{x_1}) < F^\sigma(t, \gamma_{x_1}+1)$ for all t such that $x_1 < t < x_2$ and $(t, \gamma_{x_1}+1)$ is a cell in F^σ. But now consider what happens in the insertion $k \to F^\sigma$ when we reach cell $(x_1, \gamma_{x_1}+1)$. Say we are trying to insert $\bar{k}$ into the cell. In $\bar{F}^\sigma$, the cell $(x_1, \gamma_{x_1}+1)$ contains a 0 and the only reason we cannot place $\bar{k}$ in cell $(x_1, \gamma_{x_1}+1)$ is that $\bar{k} > F^\sigma(x_1, \gamma_{x_1})$. As the insertion continues either we end up with $\bar{k}$ being inserted into cell $(x_2, \gamma_{x_1}+1)$ or the bumping path involves cells of the form $(t, \gamma_{x_1}+1)$ with $x_1 < t < x_2$. In the latter case, all the elements in the cells $(t, \gamma_{x_1}+1)$ are strictly bigger than $F^\sigma(x_1, \gamma_{x_1})$ so that by the time we reach cell $(x_2, \gamma_{x_1}+1)$, we must be trying to insert some element which is strictly bigger than $F^\sigma(x_1, \gamma_{x_1}+1)$ into cell $(x_2, \gamma_{x_1}+1)$. Thus in either case, we must be trying to insert some element z which is strictly bigger than $F^\sigma(x_1, \gamma_{x_1})$ into cell $(x_2, \gamma_{x_1}+1)$. But then $z > F^\sigma(x_1, \gamma_{x_1}) > F^\sigma(x_2, \gamma_{x_1})$ so that z cannot be inserted in $(x_2, \gamma_{x_1}+1)$ which contradicts our assumption that we placed an element on top of column x_2 in the insertion $k \to F^\sigma$. Thus the only other possibility that b and c are at the top of their columns in F^σ and we end up placing an element in cell a in the insertion $k \to F^\sigma$. But again, the B-increasing condition would force $F^\sigma(x_1, y_1) < F^\sigma(t, y_1+1)$ for all t such that $x_1 < t < x_2$ and (t, y_1+1) is a cell in F^σ. But now consider what happens in the insertion $k \to F^\sigma$ when we reach cell (x_1, y_1+1). Say we are trying to insert $\bar{k}$ into the cell. In $\bar{F}^\sigma$, the cell (x_1, y_1+1) contains a 0 and the only reason we cannot place $\bar{k}$ in cell $(x_1, \gamma_{x_1}+1)$ is that $\bar{k} > F^\sigma(x_1, y_1) = F^\sigma(b)$. As the insertion continues either we end up with $\bar{k}$ being inserted into cell (x_2, y_1+1) or the bumping path involves cells of the form (t, y_1+1) with $x_1 < t < x_2$. In the latter case, all the elements in the cells (t, y_1+1) are strictly bigger than $F^\sigma(b)$ so that by the time we reach cell $a = (x_2, y_1+1)$, we must be trying to insert some element which is strictly bigger than $F^\sigma(b)$ into cell a. Thus in either case, we must be trying to insert some element z which is strictly bigger than $F^\sigma(b)$ into cell a. But then we would have $\tilde{F}^\sigma(a) > F^\sigma(b) = \tilde{F}^\sigma(b)$ which would violate our assumption that $\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b)$.

Thus it must be the case that $\gamma_{x_1} < \gamma_{x_2}$ and a, b, c are cells in F^σ. Since the entries in the bumping path of the insertion $k \to F^\sigma$ form a strictly decreasing sequence, only one of $\{a, b, c\}$ is affected by the insertion procedure. The B-increasing condition together with the B-triple condition on $\{a, b, c\}$ in F^σ imply that $F^\sigma(b) < F^\sigma(a) \leq F^\sigma(c)$. Therefore the only entry whose replacement with a larger entry would yield the desired relative order is the entry in cell b. Note that b is not in the basement since basement elements are never bumped in the insertion $k \to F^\sigma$. Let $\underline{b}$ be the cell directly below b and $\underline{c}$ be the cell directly below c. The B-increasing condition in F^σ implies that $F^\sigma(\underline{b}) < F^\sigma(\underline{c})$. If $F^\sigma(b)$ is replaced by the entry $\tilde{F}^\sigma(b)$ where $\tilde{F}^\sigma(a) \leq \tilde{F}^\sigma(b) \leq \tilde{F}^\sigma(c)$, then we must have that $F^\sigma(b) < \tilde{F}^\sigma(b) \leq F^\sigma(\underline{b}) < F^\sigma(c)$.

We claim that is is not possible that $\bar{F}^\sigma(a) = \tilde{F}^\sigma(b)$. Let r be the least row such that $r > y_1$ and $\bar{F}^\sigma(x_2, r) < \bar{F}^\sigma(x_2, r-1)$. Then we will have the situation pictured in Figure 10 where d is the cell in column x_2 and row r. Thus all the entries of F^σ in the cells on column x_2 between a and d are equal to $\bar{F}^\sigma(a)$. Now the region of shaded cells pictured in Figure 10 are cells which are attacked or attack some cell which is equal to $\bar{F}^\sigma(a)$ and hence their entries in F^σ must all be different from $\bar{F}^\sigma(a)$. Thus $\tilde{F}^\sigma(b)$ cannot have come from any of these cells since we are assuming that $\bar{F}^\sigma(a) = \tilde{F}^\sigma(b)$. Thus $\tilde{F}^\sigma(b)$ must have come from a cell before d in reading order. But this is also impossible because $\tilde{F}^\sigma(b)$ would then meet the conditions to bump $\bar{F}^\sigma(d)$ which would violate our assumption that it bumps $F^\sigma(b)$. Thus it must be the case that $\bar{F}^\sigma(a) < \tilde{F}^\sigma(b)$.

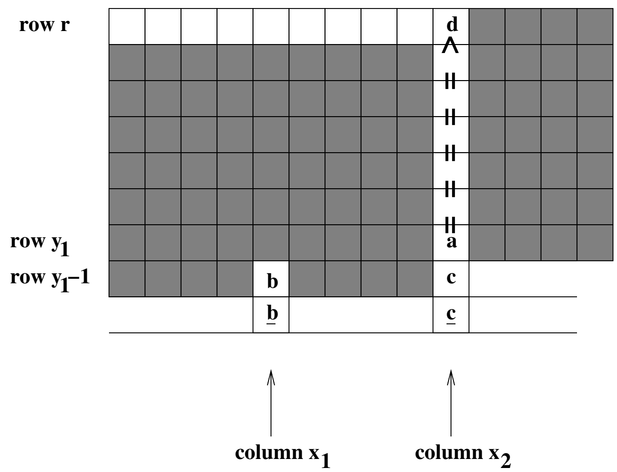


Figure 10: The cells attacked by cell equal to $\bar{F}^\sigma(a)$.

Now consider the question of where $\tilde{F}^\sigma(b)$ comes from in the insertion $k \to F^\sigma$. First $\tilde{F}^\sigma(b)$ cannot have come from a cell that precedes a in reading order because we have that $F^\sigma(a) < \tilde{F}^\sigma(b) \leq F^\sigma(c)$ so that $\tilde{F}^\sigma(b)$ meets the conditions to bump $F^\sigma(a)$. Thus $\tilde{F}^\sigma(b)$ must be in a cell which comes after a in reading order. We then have two cases.

Case 1. $\tilde{F}^\sigma(b)$ is in cell d which is in the same row as a.

In this case, we have the situation pictured in Figure 14.

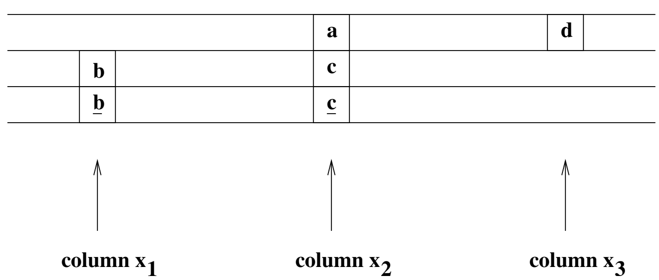


Figure 11: $\tilde{F}^\sigma(b)$ is in the same row as a.

It cannot be that $\gamma_{x_2} < \gamma_{x_3}$ because the B-increasing condition for F^σ would imply that $F^\sigma(c) < F^\sigma(d) = \tilde{F}^\sigma(b)$. Thus it must be the case that $\gamma_{x_2} \geq \gamma_{x_3}$. But then $\{a, c, d\}$ would be a coinversion type A-triple as we are assuming that $F^\sigma(a) < F^\sigma(d) = \tilde{F}^\sigma(b) \leq F^\sigma(c)$.

Case 2. $\tilde{F}^\sigma(b)$ is in the same row as b.

In this case, let $e_1, \ldots, e_s, e_{s+1} = c$ be the cells in the bumping path of the insertion of $k \to F^\sigma$ in row $y_1 - 1$, reading from left to right. Thus we are assuming that $\tilde{F}^\sigma(b) = F^\sigma(e_s)$. For each e_i, we let $\underline{e_i}$ be the cell directly below e_i and $\overline{e}_i$ be the cell directly above e_i. Thus we have the picture in Figure 15 where we are assuming that $s = 3$ and we have circled the elements in the bumping path.

Since the elements in the bumping path decrease, we have that $F^\sigma(e_1) > \cdots > F^\sigma(e_s) > F^\sigma(e_{s+1}) = \tilde{F}^\sigma(b)$ and that for each i, $F^\sigma(e_{i+1}) < F^\sigma(e_i) \leq F^\sigma(\underline{e}_{i+1})$. Let $e_j = (z_j, y_1 - 1)$. Thus $z_{s+1} = x_1$. We claim that we cannot have that $\gamma_{z_i} < \gamma_{z_{i+1}}$ for any i since the B-increasing condition would force $F^\sigma(e_{i+1}) > F^\sigma(\underline{e_i}) \geq F^\sigma(e_i)$. Thus we must have $\gamma_{z_1} \geq \cdots \geq \gamma_{z_s} \geq \gamma_{x_1}$. Similarly, we cannot have that $\gamma_{z_s} < \gamma_{x_2}$ since otherwise the B-increasing condition would force $\tilde{F}^\sigma(b) = F^\sigma(e_s) < F^\sigma(a)$. Thus we must also have $\gamma_{z_1} \geq \cdots \geq \gamma_{z_s} \geq \gamma_{x_2}$.

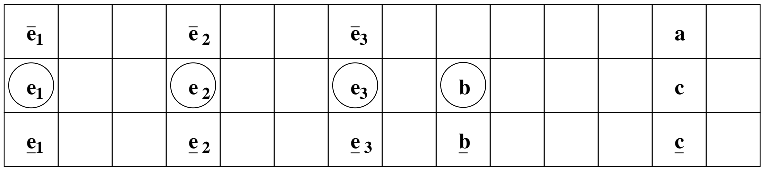


Figure 12: Picture for Subcase 2.

Now consider the A-triples $\{e_i, \underline{e}_i, c\}$ for $i = 1, \ldots, s$. We are assuming that $\tilde{F}^\sigma(b) = F^\sigma(e_s) \leq F^\sigma(c)$ so that the $\{e_s, \underline{e}_s, c\}$ A-triple condition implies that $F^\sigma(e_s) \leq F^\sigma(\underline{e}_s) < F^\sigma(c)$. But then $F^\sigma(e_s) < F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_s)$ so that the $\{e_{s-1}, \underline{e}_{s-1}, c\}$ A-triple condition also implies that $F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_{s-1}) < F^\sigma(c)$. Continuing on in this way, we can conclude that the $\{e_i, \underline{e}_i, c\}$ A-triple condition implies that $F^\sigma(e_i) \leq F^\sigma(\underline{e}_i) < F^\sigma(c)$ for each $i = 1, \ldots, s$. Next consider the A-triples $\{\overline{e}_i, e_i, a\}$ for $i = 1, \ldots, s$. We are assuming that $F^\sigma(a) < \tilde{F}^\sigma(b) = F^\sigma(e_s)$ so that the $\{\overline{e}_s, e_s, a\}$ A-triple condition implies that $F^\sigma(a) < F^\sigma(\overline{e}_s) \leq F^\sigma(e_s)$. But then $F^\sigma(e_s) < F^\sigma(e_{s-1}) \leq F^\sigma(\underline{e}_s)$ so that the $\{\overline{e}_{s-1} e_{s-1}, a\}$ A-triple condition also implies that $F^\sigma(a) < F^\sigma(\overline{e}_{s-1}) \leq F^\sigma(e_{s-1})$. Continuing on in this way, we can conclude that the $\{\overline{e}_i, e_i, a\}$ A-triple condition implies that $F^\sigma(a) < F^\sigma(\overline{e}_i) < F^\sigma(e_i)$ for each $i = 1, \ldots, s$. Thus it follows that $F^\sigma(a) < F^\sigma(e_1) \leq F^\sigma(\underline{e_1}) < F^\sigma(c)$.

Now consider the element z that bumps $F^\sigma(e_1)$ in the insertion $k \to F^\sigma$. We must have $F^\sigma(e_1) < z \leq F^\sigma(\underline{e}_1)$ so that that $F^\sigma(a) < z < F^\sigma(c)$. Thus it cannot be that $z = k$ or $z = F^\sigma(d)$ for some cell d which precedes a in reading order because that would mean that z meets the conditions to bump $F^\sigma(a)$. Thus it must be that $z = F^\sigma(d)$ for some cell d which follows a in reading order and is in the same row as a. Suppose that $d = (t, y_1)$ where $t > x_2$. Then it cannot be that $\gamma_{x_2} < \gamma_t$ since otherwise the B-increasing condition would force $F^\sigma(c) < F^\sigma(d) = z$. Thus it must be that $\gamma_{x_2} \geq \gamma_t$. But then the fact that $F^\sigma(a) < z < F^\sigma(c)$ would mean that a, c, d would be a coinversion A-triple in F^σ.

Thus we have shown that it is impossible that we bump b in this situation and, hence, there can be no coinversion B-triples in $k \to F^\sigma$. □

Lemma 5. *It F^σ is a PBF, then $\tilde{F}^\sigma = k \to F^\sigma$ satisfies the B-increasing condition.*

Proof. Suppose that F^σ is of shape $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $n \geq k$. Consider an arbitrary type B triple $\{b = (x_1, y_1), a = (x_2, y_1 + 1), c = (x_2, y_1)\}$ in $\tilde{F}^\sigma := k \to F^\sigma$ as depicted in Figure 13. Let $\overline{b}$ denote the cell immediately above b and $\underline{b}$ denote the cell immediately below b. Suppose $\tilde{F}^\sigma(b) \geq \tilde{F}^\sigma(a)$ so that the B-increasing condition fails in $\tilde{F}^\sigma$. Then there are two possibilities.

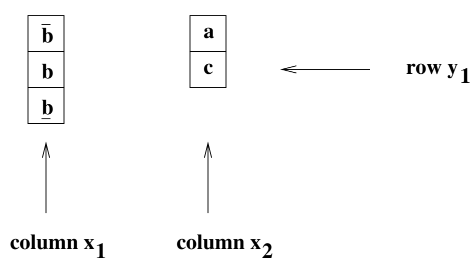


Figure 13: A type B triple.

First it could be that $\{a, b, c\}$ forms a type B triple in F^σ. This means that $\gamma_{x_1} < \gamma_{x_2}$ and hence the B-increasing condition forces $F^\sigma(b) < F^\sigma(a)$ and $F^\sigma(\underline{b}) < F^\sigma(c)$. Thus the $\{a, b, c\}$ B-triple condition implies that $F^\sigma(b) < F^\sigma(a) \leq F^\sigma(c)$ in this case. As the elements in the bumping path decrease, it must be the case that $F^\sigma(b)$ is bumped and $F^\sigma(a)$ is not bumped.

Consider the question of where $\tilde{F}^\sigma(b)$ comes from in the insertion $k \to F^\sigma$. First $\tilde{F}^\sigma(b)$ cannot have come from a cell that precedes a in reading order because we have that $F^\sigma(a) \leq \tilde{F}^\sigma(b) \leq F^\sigma(c)$ so that $\tilde{F}^\sigma(b)$ meets the conditions to bump $F^\sigma(a)$ since $F^\sigma(a) \neq \tilde{F}^\sigma(b)$ by Lemma 2. Thus $\tilde{F}^\sigma(b)$ must be in a cell which comes after a in reading order. We then have two cases.

Case 1. $\tilde{\mathrm{F}}^{\sigma}(b)$ is in cell d which is in the same row as a.

In this case, we have the situation pictured in Figure 14.

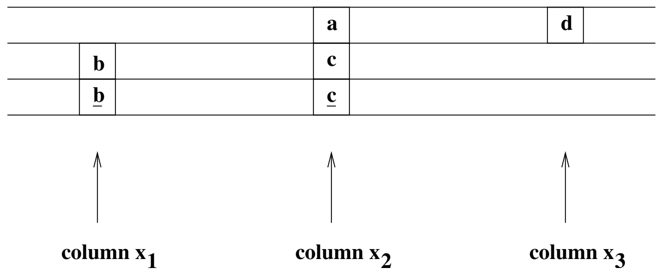


Figure 14: $\tilde{\mathrm{F}}^{\sigma}(b)$ is in the same row as a.

It cannot be that $\gamma_{x_2} < \gamma_{x_3}$ because the B-increasing condition for F^{σ} would imply that $F^{\sigma}(c) < F^{\sigma}(d) = \tilde{\mathrm{F}}^{\sigma}(b)$. Thus it must be the case that $\gamma_{x_2} \geq \gamma_{x_3}$. But then $\{a, c, d\}$ would be a coinversion type A-triple as we are assuming that $F^{\sigma}(a) < F^{\sigma}(d) = \tilde{\mathrm{F}}^{\sigma}(b) \leq F^{\sigma}(c)$.

Case 2. $\tilde{\mathrm{F}}^{\sigma}(b)$ is in the same row as b.

In this case, let $e_1, \ldots, e_s, e_{s+1} = c$ be the cells in the bumping path of the insertion of $k \to F^{\sigma}$ in row $y_1 - 1$, reading from left to right. Thus we are assuming that $\tilde{\mathrm{F}}^{\sigma}(b) = F^{\sigma}(e_s)$. For each e_i, we let $\underline{e}_i$ be the cell directly below e_i and $\overline{e}_i$ be the cell directly above e_i. Thus we have the picture in Figure 15 where we are assuming that $s = 3$ and we have circled the elements in the bumping path.

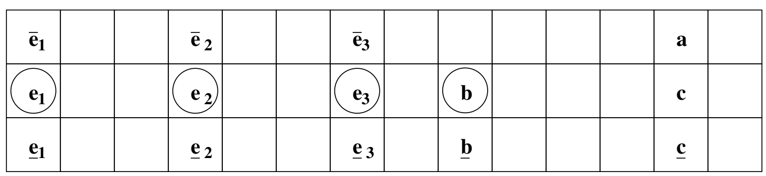


Figure 15: Picture for Subcase 2.

Since the elements in the bumping path decrease, we have that $F^{\sigma}(e_1) > \cdots > F^{\sigma}(e_s) > F^{\sigma}(e_{s+1}) = \tilde{F}^{\sigma}(b)$ and that for each i, $F^{\sigma}(e_{i+1}) < F^{\sigma}(e_i) \leq F^{\sigma}(\underline{e}_{i+1})$. Let $e_j = (z_j, y_1 - 1)$. Thus $z_{s+1} = x_1$. We claim that we cannot have that $\gamma_{z_i} < \gamma_{z_{i+1}}$ for any i since the B-increasing condition would force $F^{\sigma}(e_{i+1}) > F^{\sigma}(\underline{e_i}) \geq F^{\sigma}(e_i)$. Thus we must have $\gamma_{z_1} \geq \cdots \geq \gamma_{z_s} \geq \gamma_{x_1}$. Similarly, we cannot have that $\gamma_{z_s} < \gamma_{x_2}$ since otherwise the B-increasing condition would force $\tilde{\mathrm{F}}^{\sigma}(b) = F^{\sigma}(e_s) < F^{\sigma}(a)$. Thus we must also have $\gamma_{z_1} \geq \cdots \geq \gamma_{z_s} \geq \gamma_{x_2}$.

Now consider the A-triples $\{e_i, \underline{e}_i, c\}$ for $i = 1, \ldots, s$. We are assuming that $\tilde{\mathrm{F}}^{\sigma}(b) = F^{\sigma}(e_s) \leq F^{\sigma}(c)$ so that the $\{e_s, \underline{e}_s, c\}$ A-triple condition implies that $F^{\sigma}(e_s) \leq F^{\sigma}(\underline{e}_s) < F^{\sigma}(c)$. But then $F^{\sigma}(e_s) < F^{\sigma}(e_{s-1}) \leq F^{\sigma}(\underline{e}_s)$ so that the $\{e_{s-1}, \underline{e}_{s-1}, c\}$ A-triple condition also implies that $F^{\sigma}(e_{s-1}) \leq F^{\sigma}(\underline{e}_{s-1}) < F^{\sigma}(c)$. Continuing on in this way, we can conclude that the $\{e_i, \underline{e}_i, c\}$ A-triple condition implies that $F^{\sigma}(e_i) \leq F^{\sigma}(\underline{e}_i) < F^{\sigma}(c)$ for each $i = 1, \ldots, s$. Next consider the A-triples $\{\overline{e}_i, e_i, a\}$ for $i = 1, \ldots, s$. We are assuming that $F^{\sigma}(a) < \tilde{\mathrm{F}}^{\sigma}(b) = F^{\sigma}(e_s)$ so that the $\{\overline{e}_s, e_s, a\}$ A-triple condition implies that $F^{\sigma}(a) < F^{\sigma}(\overline{e}_s) \leq F^{\sigma}(e_s)$. But then $F^{\sigma}(e_s) < F^{\sigma}(e_{s-1}) \leq F^{\sigma}(\underline{e}_s)$ so that the $\{\overline{e}_{s-1} e_{s-1}, a\}$ A-triple condition also implies that $F^{\sigma}(a) < F^{\sigma}(\overline{e}_{s-1}) \leq F^{\sigma}(e_{s-1})$. Continuing on in this way, we can conclude that the $\{\overline{e}_i, e_i, a\}$ A-triple condition implies that $F^{\sigma}(a) < F^{\sigma}(\overline{e}_i) < F^{\sigma}(e_i)$ for each $i = 1, \ldots, s$. Thus it follows that $F^{\sigma}(a) < F^{\sigma}(e_1) \leq F^{\sigma}(\underline{e_1}) < F^{\sigma}(c)$.

Now consider the element z that bumps $F^{\sigma}(e_1)$ in the insertion $k \to F^{\sigma}$. We must have $F^{\sigma}(e_1) < z \leq F^{\sigma}(\underline{e}_1)$ so that that $F^{\sigma}(a) < z < F^{\sigma}(c)$. Thus it cannot be that $z = k$ or $z = F^{\sigma}(d)$ for some cell d which precedes a in reading order because that would mean that z meets the conditions to bump $F^{\sigma}(a)$. Thus it must be that $z = F^\sigma(d)$ for some cell d which follows a in reading order and is in the same row as a. Suppose that $d = (t, y_1)$ where $t > x_2$. Then it cannot be that $\gamma_{x_2} < \gamma_t$ since otherwise the B-increasing condition would force $F^\sigma(c) < F^\sigma(d) = z$. Thus it must be that $\gamma_{x_2} \geq \gamma_t$. But then the fact that $F^\sigma(a) < z < F^\sigma(c)$ would mean that a, c, d would be a coinversion A-triple in F^σ.

The other possibility is that $\gamma_{x_1} = \gamma_{x_2}$ and we added an element on the top of column x_2 during the insertion $k \to F^\sigma$ so that in $\tilde{F}^\sigma$, the height of column x_1 is strictly less than the height of column x_2. Now suppose that $\gamma_{x_1} = \gamma_{x_2} = y_1$ so that in the insertion $k \to F^\sigma$, the procedure is terminated by adding an element in cell a. In such a case it must be that $F^\sigma(b) = \tilde{F}^\sigma(b) \geq \tilde{F}^\sigma(a)$. Again consider the question of where $\tilde{F}^\sigma(a)$ came from. It can not be the case that $\tilde{F}^\sigma(a) = k$ or was bumped from a cell before $\bar{b}$ in the reading order because the $\tilde{F}^\sigma(a)$ could be placed on top of $F^\sigma(b)$ and $\bar{F}^\sigma(\bar{b}) = 0$ in this case. Thus $\tilde{F}^\sigma(a)$ must have been bumped from some cell d between $\bar{b}$ and a in reading order. But this is impossible since $F^\sigma(b) \geq F^\sigma(d) = \tilde{F}^\sigma(a)$ would mean that b and d do not satisfy the B-increasing condition in F^σ. Thus we must have $\gamma_{x_1} = \gamma_{x_2} > y_1$. Then the triple $\{\bar{b}, b, a\}$ is a type A triple in F^σ. We now have two possibilities, namely, either (i) $F^\sigma(\bar{b}) \leq F^\sigma(b) < F^\sigma(a)$ or (ii) $F^\sigma(a) < F^\sigma(\bar{b}) \leq F^\sigma(b)$.

If we are in case (i), then consider the type A triple $\{\underline{b}, b, c\}$ in F^σ. We cannot have that $F^\sigma(c) < F^\sigma(b) \leq F^\sigma(\underline{b})$ since then we would have $F^\sigma(a) \leq F^\sigma(c) < F^\sigma(b)$. Thus it must be the case that $F^\sigma(b) \leq F^\sigma(\underline{b}) < F^\sigma(c)$. But then $F^\sigma(b)$ must have been bumped and $F^\sigma(a)$ must not have been bumped since the elements in the bumping path form a decreasing sequence. But again this is impossible since we have

$$F^\sigma(\underline{b}) < F^\sigma(c) \leq F^\sigma(a) \leq \tilde{F}^\sigma(b)$$

which means that $\tilde{F}^\sigma(b)$ does not meet the condition to bump $F^\sigma(b)$.

If we are in case (ii), then consider the top row y_r of these columns x_1 and x_2 in F^σ. The entry in cell $e = (x_1, y_r)$ must be greater than the entry in $f = (x_2, y_r)$ by Lemma 1 as $F^\sigma(x_1, y_1+1) = F^\sigma(\bar{b}) > F^\sigma(a) = F^\sigma(x_2, y_1+1)$. Then the new entry $\tilde{F}^\sigma(x_2, y_r+1)$ added to column x_2 must have been bumped from a position after (x_1, y_r+1) in reading order, for otherwise it would be inserted on top of (x_1, y_r). But since $\tilde{F}^\sigma(x_2, y_r+1) \leq F^\sigma(x_2, y_r) < F^\sigma(x_1, y_r)$, then $\tilde{F}^\sigma(x_2, y_r+1)$ and $F^\sigma(x_1, y_r)$ would violate the B-increasing condition.

Thus we have shown that the assumption that $\tilde{F}^\sigma(b) \geq \tilde{F}^\sigma(a)$ leads to a contradiction in all cases and, hence, $\tilde{F}^\sigma$ must satisfy the B-increasing condition. □

Proposition 6. *The insertion procedure $k \to F^\sigma$ is well-defined and produces a* PBF.

Proof. Let F^σ be an arbitrary PBF of shape γ and basement $\sigma \in S_n$ and let k be an arbitrary positive integer less than or equal to n. We must show that the procedure $k \to F^\sigma$ terminates and that the resulting filling is indeed a PBF. Lemma 2 implies that at most one occurrence of any given value will be bumped to the first row. Therefore each entry i in the first row will be inserted into a column at or before the column $\sigma^{-1}(i)$. This means that the insertion procedure terminates and hence is well-defined.

Lemmas 3 and 4 imply that $k \to F^\sigma$ is a semi-standard augmented filling. Therefore $k \to F^\sigma$ is a PBF since $k \to F^\sigma$ satisfies the B-increasing condition by Lemma 5. □

Before proceeding, we make two remarks. Our first remark is concerned with the process of inverting our insertion procedure. That is, the last cell or terminal cell in the bumping path of $k \to F^\sigma$ must be a cell that originally contained 0 in $\bar{F}^\sigma$. Such a cell was not in F^σ so that the shape of $\tilde{F}^\sigma$ is the result of adding one new cell c on the top of some column of the shape of F^σ. However, there are restrictions as to where this new cell may be placed. That is, we have the following proposition which says that if c is the top cell of a sequence of consecutive columns which have the same height in $k \to F^\sigma$, then c must be in the rightmost of those columns.

Proposition 7. *Suppose that $\sigma \in S_n$ and F^σ is a* PBF *with basement σ and $k \leq n$. Suppose that F^σ has shape $\gamma = (\gamma_1, \ldots, \gamma_n)$, the shape of $k \to F^\sigma$ is $\delta = (\delta_1, \ldots, \delta_n)$, and (x, y) is the cell in δ/γ. If $x < n$, then $1 + \gamma_x \neq \gamma_{x+j}$ for $1 \leq j \leq n - x$. In particular, if $x < n$, then $\delta_x \neq \delta_{x+j}$ for $1 \leq j \leq n - x$.*

Proof. Arguing for a contradiction, suppose that $x < n$ and $1 + \gamma_x = \gamma_{x+j} = y$ for some j such that $1 \leq j \leq n - x$. Let $G^\sigma = k \to F^\sigma$ and $a = G^\sigma(x, y)$, $b = G^\sigma(x + j, y)$ and $c = G^\sigma(x, y - 1)$. Let $\bar{F}^\sigma$ and $\bar{G}^\sigma$

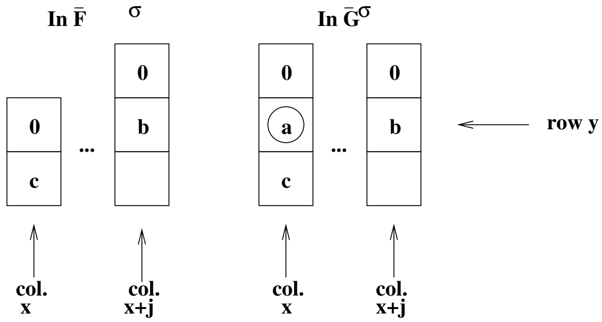


Figure 16: The end of the bumping path in $k \to F^\sigma$.

be the fillings which result by placing 0's on top of the columns of F^σ and G^σ respectively. Thus we would have this situation pictured in Figure 16 for the end of the bumping path in the insertion $k \to F^\sigma$.

Hence b is at the top of column $x+j$ in both F^σ and G^σ and neither b nor c is bumped during the insertion of $k \to F^\sigma$. Note that the B-increasing condition in F^σ forces that $c < b$. The $\{(x,y),(x+j,y),(x,y-1)\}$ A-triple condition in G^σ implies that

$$G^\sigma(x,y) = a \le G^\sigma(x,y-1) = c < G^\sigma(x+j,y) = b.$$

Now consider the question of where a came from in the bumping path of the insertion $k \to F^\sigma$. It cannot be that $a = k$ or a was bumped from a cell before $(x+j,y+1)$ because of the fact that $a < b$ would allow a to be inserted on top of b. Thus either (i) $a = F^\sigma(z,y+1)$ for some $z > x+j$ or (ii) $a = F^\sigma(z,y)$ for some $z < x$. Case (i) is impossible since then we would have $\gamma_{x+j} < \gamma_z$ and the B-increasing condition in F^σ would force $b < a$.

If case (ii) holds, let $e_1,\ldots,e_s,e_{s+1} = (x,y)$ be the cells in the bumping path of the insertion of $k \to F^\sigma$ in row y, reading from left to right. Thus we are assuming that $a = F^\sigma(e_s)$. For each e_i, we let $\underline{e}_i$ be the cell directly below e_i. Thus we have the picture in Figure 17 where we are assuming that $s = 3$ and we have circled the elements in the bumping path.

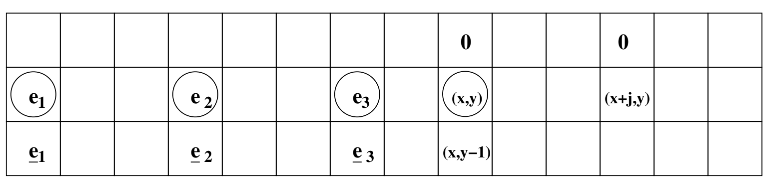


Figure 17: Picture for case (ii).

Since the elements in the bumping path decrease, we have that $F^\sigma(e_1) > \cdots > F^\sigma(e_s) = a$ and that for each i, $F^\sigma(e_{i+1}) < F^\sigma(e_i) \le F^\sigma(\underline{e}_{i+1})$. Let $e_j = (z_j,y)$. Thus $z_{s+1} = x$. We cannot have $\gamma_{z_i} < \gamma_{z_{i+1}}$ for any i since the B-increasing condition in F^σ would force $F^\sigma(e_{i+1}) > F^\sigma(\underline{e_i}) \ge F^\sigma(e_i)$. Thus we must have $\gamma_{z_1} \ge \cdots \ge \gamma_{z_s}$. Since c is at the top of its column in F^σ, we also have that $\gamma_{z_s} > \gamma_x = \gamma_{z_{s+1}}$. Thus $\gamma_{z_1} \ge \cdots \ge \gamma_{z_s} > \gamma_x$.

Now consider the A-triples $\{e_i,\underline{e}_i,(x+j,y)\}$ for $i = 1,\ldots,s$ in F^σ. We have established that $F^\sigma(e_s) = a \le c < b = F^\sigma(x+j,y)$. Thus it follows from the $\{e_s,\underline{e}_s,(x+j,y)\}$ A-triple condition that $F^\sigma(e_s) \le F^\sigma(\underline{e}_s) < b$. But then $F^\sigma(e_s) < F^\sigma(e_{s-1}) \le F^\sigma(\underline{e}_s)$ so that the $\{e_{s-1},\underline{e}_{s-1},(x+j,y)\}$ A-triple condition also implies that $F^\sigma(e_{s-1}) \le F^\sigma(\underline{e}_{s-1}) < b$. Continuing on in this way, we can conclude from the $\{e_i,\underline{e}_i,(x+j,y)\}$ A-triple condition that $F^\sigma(e_i) \le F^\sigma(\underline{e}_i) < b$ for $i = 1,\ldots,s$.

Now consider the element z that bumps $F^\sigma(e_1)$ in the insertion $k \to F^\sigma$. We must have $F^\sigma(e_1) < z \le F^\sigma(\underline{e}_1) < b$. Thus it cannot be that $z = k$ or $z = F^\sigma(d)$ for some cell d which precedes $(x+j,y+1)$ in reading order because that would mean that z meets the conditions to be placed on top of b. Thus it must be that $z = F^\sigma(d)$ for some cell d which follows $(x+j,y+1)$ in reading order. Suppose that $d = (t,y+1)$ where $t > x+j$. We are assuming that $(x+j,y)$ is the top cell in column $x+j$. Thus it must be the case that $\gamma_{x+j} < \gamma_t$. But then the B-increasing condition in F^σ would force $b < z$ which is a contradiction. Thus case (ii) cannot hold either which implies $1 + \gamma_x \neq \gamma_{x+j}$. ☐

Except for the restrictions determined by Proposition 7, we can invert the insertion procedure. That is, to invert the procedure $k \to F^\sigma$, begin with the entry r_j contained in the new cell appended to F^σ and read backward through the reading order beginning with this cell until an entry is found which is greater than r_j and immediately below an entry less than or equal to r_j. Let this entry be r_{j-1}, and repeat. When the first cell of $k \to F^\sigma$ is passed, the resulting entry is $r_1 = k$ and the procedure has been inverted.

Our second remark concerns the special case where $\sigma = \bar{\epsilon}_n$ and $k \leq n$. Then we claim that our insertion procedure is just a twisted version of the usual RSK row insertion algorithm. That is, we know that F^σ must be of shape $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ and that F^σ is weakly decreasing in columns, reading from bottom to top, and is strictly decreasing in rows, reading from left to right. Now if $k \leq F^\sigma(1, \gamma_1)$, then we just add k to the top of column 1 to form $k \to F^\sigma$. Otherwise suppose that $F^\sigma(1, y_1) \geq k > F^\sigma(1, y_1+1)$. Then all the elements in $\bar{F}^\sigma$ that lie weakly above row $y_1 + 1$ and strictly to the right of column 1 must be less than or equal to $F^\sigma(1, y_1 + 1)$. Thus the first place that we can insert k is in cell $(1, y_1 + 1)$. Thus it will be that case that k bumps $F^\sigma(1, y_1 + 1)$. Since elements in the bumping path are decreasing and all the elements in column 1 below row $y_1 + 1$ are strictly larger than $F^\sigma(1, y_1 + 1)$, it follows that none of them can can be involved in the bumping path of the insertion $k \to F^\sigma$. It is then easy to check that since $F^\sigma(1, y_1 + 1) \leq n - 1$, the result of the insertion $k \to F^\sigma$ is the same as the result of the insertion of $F^\sigma(1, y_1 + 1)$ into the PBF formed from F^σ by removing the first column and then adding back column 1 of F^σ with $F^\sigma(1, y_1 + 1)$ replaced by k. Thus our insertion process satisfies the usual recursive definition of the RSK row insertion algorithm. Hence, in the special case where the basement permutation is $\bar{\epsilon}_n$ and $k \leq n$, our insertion algorithm is just the usual RSK row insertion algorithm subject to the condition that we have weakly decreasing columns and strictly decreasing rows instead of strictly increasing columns and weakly increasing rows.

4 General Properties of the insertion algorithm

In this section, we shall prove several fundamental properties of the insertion algorithm $k \to F^\sigma$. In particular, our results in this section will allow us to prove that our insertion algorithm can be factored through the twisted version of RSK row insertion described in the previous section.

Let s_i denote the transposition $(i, i+1)$ so that if $\sigma = \sigma_1 \ldots \sigma_n$, then

$$s_i \sigma = \sigma_1 \ldots \sigma_{i-1} \sigma_{i+1} \sigma_i \sigma_{i+2} \ldots \sigma_n.$$

Our next lemma will describe the difference between inserting a word w into E^σ versus inserting w into $E^{s_i \sigma}$. Recall that E^σ is the empty filling which just consists of the basement whose entries are $\sigma_1, \ldots, \sigma_n$ reading from left to right. If $w = w_1 \ldots w_t$, then we let

$$w \to E^\sigma = w_t \to (\ldots (w_2 \to (w_1 \to E^\sigma)) \ldots).$$

Theorem 8. *Let w be an arbitrary word whose letters are less than or equal to n and suppose that $\sigma = \sigma_1 \ldots \sigma_n$ is a permutation such that $\sigma_i < \sigma_{i+1}$. Let $F^\sigma = w \to E^\sigma$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the shape of F^σ, $F^{s_i \sigma} = w \to E^{s_i \sigma}$ and $\delta = (\delta_1, \ldots, \delta_n)$ be the shape of $F^{s_i \sigma}$. Then*

1. *$\{\gamma_i, \gamma_{i+1}\} = \{\delta_i, \delta_{i+1}\}$ and $\delta_i \geq \delta_{i+1}$,*
2. *$F^{s_i \sigma}(j, k) = F^\sigma(j, k)$ for $j \neq i, i+1$ so that $\gamma_j = \delta_j$ for all $j \neq i, i+1$,*
3. *for all j, $\{F^{s_i \sigma}(i, j), F^{s_i \sigma}(i+1, j)\} = \{F^\sigma(i, j), F^\sigma(i+1, j)\}$, and*
4. *$F^{s_i \sigma}(i, j) > F^{s_i \sigma}(i+1, j)$, where we let $F^{s_i \sigma}(i+1, j) = 0$ if $(i+1, j)$ is not a cell in F^σ.*

Proof. Before starting our proof, we pause to make an observation. Namely, since $F^{s_i \sigma}$ is a PBF and $F^{s_i \sigma}(i, 0) = \sigma_{i+1} > \sigma_i = F^{s_i \sigma}(i+1, 0)$, Lemma 1 implies that $F^{s_i \sigma}(i, j) > F^{s_i \sigma}(i+1, j)$ for all j if $\delta_i \geq \delta_{i+1}$. Thus condition (1) will automatically imply condition (4).

We proceed by induction on the length of w. If w consists of a single letter w_1, then in $w_1 \to E^{\sigma}$, there there will be a σ_s in the basement such that w_1 is placed on top of σ_s since we are assuming $w_1 \leq n$. We then have three cases.

Case 1. $s \in \{1, \ldots, i-1\}$.
In this case w_1 will be placed on top of σ_s in $w_1 \to E^{s_i\sigma}$ so that $F^{\sigma} = F^{s_i\sigma}$.

Case 2. $s \in \{i+2, \ldots, n\}$.
In this case it must be the case that $w_1 > \sigma_j$ for all $j \leq i+1$. Hence it will still be the case that w_1 will be placed on top of σ_s in $w_1 \to E^{s_i\sigma}$ so that $F^{\sigma} = F^{s_i\sigma}$.

Case 3. $s \in \{i, i+1\}$
In this case it must be that $w_1 > \sigma_j$ for all $j \leq i-1$ and either $w_1 \leq \sigma_i$ in which case w_1 will be placed on top of σ_i in $w_1 \to E^{\sigma}$ or $\sigma_{i+1} \geq w_1 > \sigma_i$ in which case w_1 will be placed on top of σ_{i+1} in $w_1 \to E^{\sigma}$. However, in either case, $w_1 \leq \sigma_{i+1}$ so that w_1 will be placed on top of $(s_i\sigma)_i = \sigma_{i+1}$ in $w_1 \to E^{s_i\sigma}$.

It is then easy to check that conditions (1)-(4) hold in each of the three cases.

Now suppose that the theorem holds for all words of length less than t. Then let $G = w_1 \ldots w_{t-1} \to E^{\sigma}$ and $H = w_1 \ldots w_{t-1} \to E^{s_i\sigma}$ and suppose G has shape $\alpha = (\alpha_1, \ldots, \alpha_n)$ and H has shape $\beta = (\beta_1, \ldots, \beta_n)$. Let $\bar{G}$ and $\bar{H}$ be the fillings with 0's added to the tops of the columns of G and H respectively. Let $\tilde{G} = w_t \to G$ and $\tilde{H} = w_t \to H$ and suppose that $\tilde{G}$ has shape $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\tilde{H}$ has shape $\delta = (\delta_1, \ldots, \delta_n)$. We compare the bumping path of $w_t \to H$ to the bumping path in $w_t \to G$. That is, in the insertion process $w_t \to H$, suppose we come to a point were we are inserting some element c which is either w_t or some element bumped in the insertion $w_t \to H$ into the cells (i,j) and $(i+1,j)$. Assume by induction, that the insertion of $w_t \to G$ will also insert c into the cells (i,j) and $(i+1,j)$. This will certainly be true the first time the bumping paths interact with elements in columns i and $i+1$ since our induction assumption ensures that $\bar{G}$ restricted to columns $1, \ldots, i-1$ equals $\bar{H}$ restricted to columns $1, \ldots, i-1$. We let $x = \bar{H}(i,j)$, $y = \bar{H}(i+1,j)$, $\underline{x} = \bar{H}(i,j-1)$, and $\underline{y} = \bar{H}(i+1,j-1)$. Our inductive assumption implies that if $x > 0$, then $x > y$ and if $\underline{x} > 0$, then $\underline{x} > \underline{y}$. Our goal is to analyze how the insertion of c interacts with elements in cells (i,j) and $(i+1,j)$ during the insertions $w_t \to H$ and $w_t \to G$. We will show that either

($\mathbb{A}$) the bumping path does not interact with cells (i,j) and $(i+1,j)$ during both the insertions $w_t \to H$ and $w_t \to G$,

($\mathbb{B}$) the insertion of c into cells (i,j) and $(i+1,j)$ results in inserting some c' into the next cell in reading order after $(i+1,j)$ in both $w_t \to H$ and $w_t \to G$, or

($\mathbb{C}$) both insertions end up terminating in one of (i,j) or $(i+1,j)$.

This will ensure that $w_t \to H$ and $w_t \to G$ are identical outside of columns i and $i+1$ thus proving condition (2) of the lemma. We now consider several cases.

Case I. $x = y = 0$.
This means that x and y sit on top of columns i and $i+1$ respectively in H. Since $\{\bar{G}(i,j), \bar{G}(i+1,j)\} = \{\bar{H}(i,j), \bar{H}(i+1,j)\}$, it must be that x and y also sit on top of columns i and $i+1$ in G also. We also know that either (a) $G(i,j-1) = \underline{x}$ and $G(i+1,j-1) = \underline{y}$ or (b) $G(i,j-1) = \underline{y}$ and $G(i,j-1) = \underline{x}$.

First suppose that $c \leq \underline{x}$. Then in $w_t \to H$, the insertion will terminate by putting c on top of $\underline{x}$. This will ensure that in $\tilde{H}$, the length of column i will be greater than the length of column $i+1$. In case (a), the insertion $w_t \to G$ will terminate by placing c on top of $\underline{x}$ and in case (b) the insertion $w_t \to G$ will terminate by placing c on top of $\underline{y}$ if $c \leq \underline{y}$, or by placing c on top of $\underline{x}$ if $\underline{y} < c \leq \underline{x}$. In either case, it is easy to see that conditions (1)-(4) will hold.

Next suppose that $\underline{x} < c$, Then in $w_t \to H$, c will not be placed in either cell (i,j) or $(i+1,j)$ so that the result is that c will end up being inserted in the next cell in reading order after $(i+1,j)$. But clearly in both cases (a) and (b), c will not be placed in either cell (i,j) or $(i+1,j)$ in the insertion $w_t \to G$ so that the result is that c will end up being inserted in the next cell in reading order after $(i+1,j)$.

Case II. $x > 0$, $y = 0$, and $\underline{y} > 0$.
In this case, $\underline{y}$ is at the top of a column in H. We have four possible cases in $\bar{G}$, namely,
(A) $\bar{G}(i,j) = x, \bar{G}(i+1,j) = 0, \bar{G}(i,j-1) = \underline{x}$ and $\bar{G}(i+1,j-1) = \underline{y}$,

(B) $\bar{G}(i,j) = 0, \bar{G}(i+1,j) = x, \bar{G}(i,j-1) = \underline{x}$ and $\bar{G}(i+1,j-1) = \underline{y}$,
(C) $\bar{G}(i,j) = x, \bar{G}(i+1,j) = 0, \bar{G}(i,j-1) = \underline{y}$ and $\bar{G}(i+1,j-1) = \underline{x}$,
(D) $\bar{G}(i,j) = 0, \bar{G}(i+1,j) = x, \bar{G}(i,j-1) = \underline{y}$ and $\bar{G}(i+1,j-1) = \underline{x}$.
These four cases are pictured in Figure 18

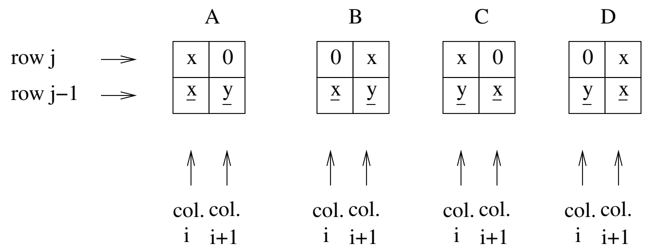


Figure 18: Possibilities in $\bar{G}$.

First consider the case where in the insertion $w_t \to H$, c does not bump x and the insertion terminates with c being placed on top of $\underline{y}$. Then we know that $\delta_i \geq \delta_{i+1}$ and $\tilde{H}$ is a PBF so that it must be that case that $x > c$ by Lemma 1 and, hence, c does not bump x in the insertion $w_t \to G$ in either case (A) or (C). Since c can be placed on top of $\underline{y}$, we have $c \leq \underline{y} < \underline{x}$. Thus c cannot bump x and can be placed on top of either $\underline{x}$ or $\underline{y}$. It follows that in cases (A) and (C) the insertion $w_t \to G$ must terminate by placing c in cell $(i+1,j)$ and in cases (B) and (D) the insertion $w_t \to G$ must terminate by placing c in cell (i,j). It follows that in all cases cells (i,j) and $(i+1,j)$ will contain c and x in some order in $\tilde{G}$ which means that (1) -(4) will hold in this case.

Next consider the case where in the insertion $w_t \to H$, c bumps x and x terminates the insertion by being placed on top of $\underline{y}$. Then clearly we will have $\delta_i \geq \delta_{i+1}$. We also know that $x < c \leq \underline{x}$ and $x \leq \underline{y}$. This rules out case (D) since otherwise $\alpha_i < \alpha_{i+1}$ and $\{G(i,j-1) = \underline{y}, G(i+1,j-1) = \underline{x}, G(i+1,j) = x\}$ would violate the B-triple condition for G. Now if G satisfies case (A), then G and H agree on the entries in $\{(i,j),(i+1,j),(i,j-1),(i+1,j-1)\}$ so that in the insertion $w_t \to G$, c will also bump x and x will be placed on top of $\underline{y}$. In case (B), c will just be inserted on top of $\underline{x}$. In case (C), c will bump x if $c \leq \underline{y}$ and otherwise c will just be inserted on top of $\underline{x}$. Thus in all three cases $\{\bar{G}(i,j), \bar{G}(i+1,j)\} = \{x, c\}$ so that condition (1)-(4) will hold in this case.

Next consider the case where in the insertion $w_t \to H$, c bumps x and x cannot be placed on top of $\underline{y}$ so that x is inserted in the next cell in reading order after $(i+1,j)$. Then we must have $\underline{y} < x < c \leq \underline{x}$. This rules out cases (B) and (C) since x cannot sit on top of $\underline{y}$. It is also the case that c cannot sit on top of $\underline{y}$ so that it is easy to see that in both cases (A) and (D), c will bump x and the result is that x will be inserted in the next cell in reading order after $(i+1,j)$.

Finally consider the case where c does not bump x and c cannot be placed on top of $\underline{y}$ in the insertion $w_t \to H$ so that c is inserted in the next cell in reading order after $(i+1,j)$. The fact that c does not bump x means that either $c > \underline{x}$ or $c \leq x$. The fact that c cannot be placed on top of $\underline{y}$ means that $c > \underline{y}$. If $c > \underline{x} > \underline{y}$, then it is clear that in cases (A)-(D), c does not meet the conditions for the entries in cells (i,j) and $(i+1,j)$ to change so that so that the result is that c will be inserted in the next cell in reading order after $(i+1,j)$. If $c \leq x$, then we have $\underline{y} < c \leq x$. Thus x cannot sit on top of $\underline{y}$ so that cases (B) and (C) are ruled out. Case (A) is the same as in H. However in case (D), we still have that c cannot sit on top of $\underline{y}$ and c cannot bump x so that so that the result is that c will be inserted in the next cell in reading order after $(i+1,j)$.

Case III. $x, \underline{x} > 0$ and $y = \underline{y} = 0$.
In this case, there are no cells in H in positions $(i+1,j)$ and $(i+1,j-1)$. But this means that either G is identical to H or there are no cells in G in positions (i,j) and $(i,j-1)$ and $G(i+1,j) = x$ and $G(i+1,j-1) = \underline{x}$. Thus, in this case as far as the insertion of c into cells (i,j) and $(i+1,j)$, we see the same possibilities for bumping in both the insertions $w_t \to H$ and $w_t \to G$. Thus either c will bump x and x will be inserted in the cell following $(i+1,j)$ in reading order in both the insertions $w_t \to H$ and $w_t \to G$ or c will not bump x and c will be inserted in the cells following $(i+1,j)$ in reading order in both the insertions

$w_t \to H$ and $w_t \to G$.

Case IV. $x, \underline{x}, y, \underline{y} > 0$.
In this case, we know that (i,j), $(i+1,j)$, $(i,j-1)$ and $(i+1,j-1)$ are all cells in G. Once again we have four possible cases in G, namely,
(A) $G(i,j) = x, G(i+1,j) = y, G(i,j-1) = \underline{x}$ and $G(i+1,j-1) = \underline{y}$,
(B) $G(i,j) = y, G(i+1,j) = x, G(i,j-1) = \underline{x}$ and $G(i+1,j-1) = \underline{y}$,
(C) $G(i,j) = x, G(i+1,j) = y, G(i,j-1) = \underline{y}$ and $G(i+1,j-1) = \underline{x}$,
(D) $G(i,j) = y, G(i+1,j) = x, G(i,j-1) = \underline{y}$ and $G(i+1,j-1) = \underline{x}$.

These are pictured in Figure 19.

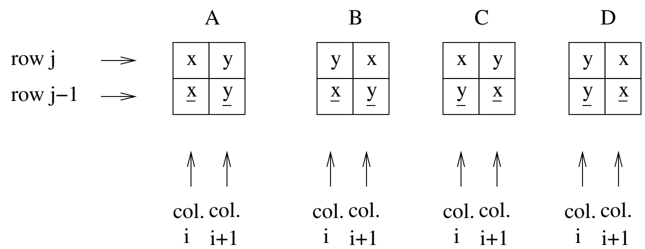


Figure 19: Possibilities in G.

Note that if we are in case (A), then the same thing will happen in both the insertions $w_t \to H$ and $w_t \to G$ as c is inserted into cell (i,j) and c' emerges from cell $(i+1,j)$ to be inserted into the cell following $(i+1,j)$ in reading order so there is nothing to prove in this case.

First suppose that in the insertion $w_t \to H$, c bumps x, but x does not bump y so that the result is that x will be inserted into the cell following $(i+1,j)$ in reading order. Since $y < x$, the reason that x does not bump y must be that $x > \underline{y}$. Thus it must be the case $\underline{x} \geq x > \underline{y} \geq y$. This means that cases (B) and (C) are impossible since x cannot sit on top of $\underline{y}$ in G. But then $c > x > \underline{y}$ so that in the insertion $w_t \to G$, c cannot bump y in case (D). Thus in case (D), c will bump x so that the result is that x will be inserted into the cell following $(i+1,j)$ in reading order as desired.

Next consider the case where c does not bump x but bumps y. Since c does not bump x then we either have (i) $c > \underline{x}$ or (ii) $c \leq x$. If (i) holds, then $c > \underline{x} > \underline{y}$ so once again c cannot bump y. Thus (ii) must hold. Since c bumps y, we have that $y < c \leq \underline{y}$. Thus we have two possibilities, namely, $y < c \leq \underline{y} < x$ or $y < c \leq x \leq \underline{y} \leq x$. First suppose that $y < c \leq \underline{y} < x$. Then cases B and C are impossible since x cannot sit on top of $\underline{y}$. In case D, c will bump y but y cannot bump x since $y < x$ so that y is inserted in the next cell after $(i+1,j)$. Next suppose that $y < c \leq x \leq \underline{y} \leq x$. Then in case B, c will bump but y cannot bump x since $y < x$, so that y is inserted in the next cell after $(i+1,j)$. In case C, c does not bump x since $c \leq x$ so that c will bump y and y will be inserted in the next cell after $(i+1,j)$. In case D, c will bump y but y cannot bump x since $y < x$, so that y is inserted in the next cell after $(i+1,j)$. Thus in every case, y will be inserted in the next cell after $(i+1,j)$.

Next consider the case where in the insertion $w_t \to H$, c bumps x, and then x bumps y so that the result is that y will be inserted into the cell following $(i+1,j)$ in reading order. In this case we must have $y < x \leq \underline{y} < \underline{x}$ and $x < c \leq \underline{x}$. In case (B), it is easy to see that in the insertion $w_t \to G$, c will bump y since $y < x < c \leq \underline{x}$, but y will not bump x so that the result is that y will be inserted into the cell following $(i+1,j)$ in reading order. In case (C), c will bump x and then x will bump y if $c \leq \underline{y}$. However if $c > \underline{y}$, then c will not bump x but it will bump y. Thus in either situation, the result is that y will be inserted into the cell following $(i+1,j)$ in reading order. Finally consider case (D). If $c \leq \underline{y}$, then c will bump y but y will not bump x so that again the result is that y will be inserted into the cell following $(i+1,j)$ in reading order. Now if $c > \underline{y}$, then we must have that $y < x \leq \underline{y} < c \leq \underline{x}$. We claim that this is impossible. Recall that α_i and α_{i+1} are the heights of column i and $i+1$ in G, respectively. Now if $\alpha_i \geq \alpha_{i+1}$, then $\{G(i,j) = y, G(i,j-1) = \underline{y}, G(i+1,j) = x\}$ would be a type A coinversion triple in G and if $\alpha_i < \alpha_{i+1}$, then $\{G(i,j-1) = \underline{y}, G(i+1,j) = x, G)i+1,j-1) = \underline{x}\}$ would be a type B coinversion triple in G.

Finally consider the case where c does not bump either x or y in the insertion $w_t \to H$ so that c is inserted into the cells following $(i+1,j)$ in reading order. Then either $c < y < x$ so that c cannot bump either x or y in cases (B)-(D) or $c > \underline{x} > \underline{y}$ so again c cannot bump either x or y in cases (B)-(D). Thus in all cases, the result is that c will be inserted into the cells following $(i+1,j)$ in reading order.

Thus we have shown that conditions ($\mathbb{A}$), ($\mathbb{B}$), and ($\mathbb{C}$) always holds which, in turn, implies that conditions (1)-(4) always hold. $\square$

Before we proceed, we pause to make one technical remark which will be important for our results in Section 5. That is, a careful check of the proof of Theorem 8 will show that we actually proved the following.

Corollary 9. *Suppose that $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, $\sigma_i < \sigma_{i+1}$, and $w = w_1 \ldots w_t \in \{1,\ldots,n\}^n$. For $j = 1,\ldots,t$, let $F_j^{\sigma} = w_1 \ldots w_j \to E^{\sigma}$ and $F_j^{s_i\sigma} = w_1 \ldots w_j \to E^{s_i\sigma}$. Let $F_0^{\sigma} = E^{\sigma}$ and $F_0^{s_i\sigma} = E^{s_i\sigma}$. Let $\alpha^{(j)}$ be the shape of F_j^{σ} and $\beta^{(j)}$ be the shape of $F_j^{s_i\sigma}$. Then for all $i \geq 1$, the cells in $\alpha^{(i)}/\alpha^{(i-1)}$ and $\beta^{(i)}/\beta^{(i-1)}$ lie in the same row.*

Proof. It is easy to prove the lemma by induction on t. The lemma is clearly true for $t = 1$ since inserting w_1 into either E^{σ} or $E^{s_i\sigma}$ will create a new cell in the first row. Then it is easy to check that our proof of Theorem 8 establishing properties ($\mathbb{A}$), ($\mathbb{B}$), and ($\mathbb{C}$) for the insertions $w_t \to F_{t-1}^{\sigma}$ and $w_t \to F_{t-1}^{s_i\sigma}$ implies that the cells in $\alpha^{(t)}/\alpha^{(t-1)}$ and $\beta^{(t)}/\beta^{(t-1)}$ must lie in the same row. $\square$

For any alphabet A, we let A^* denote the set of all words over the alphabet A. If $w \in \{1,\ldots,n\}^*$, then we let $P^{\sigma}(w) = w \to E^{\sigma}$, which we call the σ-insertion tableau of w and let $\gamma^{\sigma}(w) = (\gamma_1^{\sigma}(w),\ldots,\gamma_n^{\sigma}(w))$ be the composition corresponding to the shape of $P^{\sigma}(w)$. Theorem 8 has a number of fundamental consequences about the set of σ-insertion tableaux of w as σ varies over the symmetric group S_n. Note that $P^{\bar{\epsilon}_n}(w)$ arises from w by performing a twisted version of the RSK row insertion algorithm. Hence $\gamma^{\bar{\epsilon}_n}(w)$ is always a partition.

Then we have the following corollary.

Corollary 10. *Suppose that $w \in \{1,\ldots,n\}^*$.*

1. *$P^{\sigma}(w)$ is completely determined by $P^{\epsilon_n}(w)$.*
2. *For all $\sigma \in S_n$, $\gamma^{\sigma}(w)$ is a rearrangement of $\gamma^{\bar{\epsilon}_n}(w)$).*
3. *For all $\sigma \in S_n$, the set of elements that lie in row j of $P^{\sigma}(w)$ equals the set of elements that lie in row j of $P^{\bar{\epsilon}_n}(w)$ for all $j \geq 1$.*
4. *For all $\sigma \in S_n$ and all $1 \leq i < j \leq n$, if $\sigma_i > \sigma_j$, then $\gamma_i^{\sigma}(w) \geq \gamma_j^{\sigma}(w)$ and for all s, $P^{\sigma}(w)(i,s) > P^{\sigma}(w)(j,s)$.*

Proof. For (1) and (4), note that if $\sigma = \sigma_1 \ldots \sigma_n$ where $\sigma_i < \sigma_{i+1}$, then Theorem 8 tells us that $P^{\sigma}(w)$ completely determines $P^{s_i\sigma}(w)$. That is, to obtain $P^{s_i\sigma}(w)$ from $P^{\sigma}(w)$ we need only ensure that when both (i,j) and $(i+1,j)$ are cells in $P^{\sigma}(w)$, then the elements in those two cells in $P^{\sigma}(w)$ must be arranged in decreasing order in $P^{s_i\sigma}(w)$. If only one of the cells (i,j) and $(i+1,j)$ is in $P^{\sigma}(w)$, then the element in the cell that is occupied in $P^{\sigma}(w)$ must be placed in cell (i,j) in $P^{s_i\sigma}(w)$. Moreover, it is clear that if (4) holds for $P^{\sigma}(w)$, then (4) must also hold for $P^{s_i\sigma}(w)$. Since we can get from ϵ_n to any $\sigma \in S_n$ by applying a sequence of adjacent transpositions where we increase the number of inversions at each step, it follows that $P^{\sigma}(w)$ is completely determined by $P^{\epsilon_n}(w)$. Note also that (4) holds for $P^{\epsilon_n}(w)$ automatically so that property (4) must also hold for all $P^{\sigma}(w)$.

For (2) and (3), note that for any $\sigma \in S_n$, we can get from σ to $\bar{\epsilon}_n$ by applying a sequence of adjacent transpositions where we increase the number of inversions at each step. Thus it follows from Theorem 8 that the set of column heights in $\gamma^{\bar{\epsilon}_n}(w)$ must be a rearrangement of the set of column heights of $\gamma^{\sigma}(w)$.

Moreover, it also follows that the set of elements in row j of $P^{\bar{\epsilon}_n}(w)$ must be the same as the set of elements in row j of $P^{\sigma}(w)$. Note that all the elements in a row of a PBF must be distinct by the non-attacking properties of a PBF. $\square$

Mason [8] introduced a shift map ρ which takes any PBF F with basement equal to ϵ_n to a reverse row strict tableau $\rho(F)$ by simply putting the elements which appear in row j of F (where $j \geq 1$) in decreasing order in row j of $\rho(F)$, reading from left to right. This map is pictured at the top of Figure 20. We can then add a basement below $\rho(F)$ which contains the permutation $\bar{\epsilon}_n$ to obtain a PBF with basement equal to $\bar{\epsilon}_n$.

We can extend this map to PBFs with an arbitrary basement σ in a natural way. That is, if F^σ is a PBF with basement $\sigma \in S_n$, let $\rho_\sigma(F^\sigma)$ be the PBF with basement $\bar{\epsilon}_n$ by simply putting the elements which appear in row j of F^σ in decreasing order in row j of $\rho_\sigma(F^\sigma)$ for $j \geq 0$, reading from left to right. This map is pictured at the bottom of Figure 20. To see that $\rho_\sigma(F^\sigma)$ is a reverse row strict tableau, we need only check that $\rho_\sigma(F^\sigma)$ is weakly decreasing in columns from bottom to top. But this property is an immediate consequence of the fact that every element in row j of F^σ where $j \geq 1$ is less than or equal to the element it sits on top of in F^σ.

Mason [8] showed that for any reverse row strict tableaux T, there is a unique PBF F_T with basement equal to ϵ_n such that $\rho(F_T) = T$. Thus ρ^{-1} is uniquely defined. In fact, Mason gave a procedure for constructing $\rho^{-1}(T)$. That is, assume that T has k rows and that P_i is the set of elements of T that lie in row i.

Definition of ρ^{-1} [8]. Inductively assume that the first i rows of T, $\{P_1, \ldots, P_{i-1}\}$ have been mapped to a PBF $F^{(i-1)}$ with basement ϵ_n in such a way the elements in row j of $F^{(i-1)}$ are equal to P_j for $j = 1, \ldots, i-1$. Let $P_i = \{\alpha_1 > \alpha_2 > \cdots > \alpha_{s_i}\}$. There exists an element greater than or equal to α_1 in row $i-1$ since α_1 sits on top of some element in T. Place α_1 on top of the left-most such element in row $i-1$ of $F^{(i-1)}$. Next assume that we have placed $\alpha_1, \ldots, \alpha_{k-1}$. Then there are at least k elements of P_{i-1} that are greater than or equal to α_k since each of $\alpha_1, \ldots, \alpha_k$ sit on top of some element in row $i-1$ of T. Place α_k on top of the left-most element in row $i-1$ of $F^{(i-1)}$ which is greater than or equal to α_k which does not have one of $\alpha_1, \ldots, \alpha_{k-1}$ on top of it.

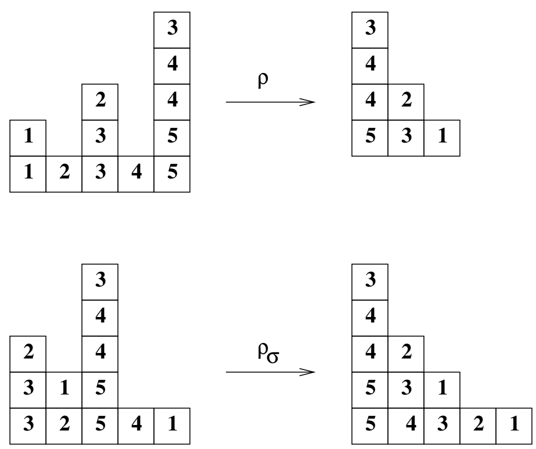


Figure 20: The ρ and ρ_σ maps.

Now suppose that $w \in \{1, \ldots, n\}^*$. We let $rr(w)$ be the word that results by reading the cells of $P^{\bar{\epsilon}_n}(w)$ in reverse reading order. Thus $rr(w)$ is just the word which consists of the elements in the first row of $P^{\bar{\epsilon}_n}(w)$ in increasing order, followed by the elements in the second row of $P^{\bar{\epsilon}_n}(w)$ in increasing order, etc. For example, if $w = 1\ 3\ 2\ 4\ 3\ 2\ 1\ 4$, then $P^{\bar{\epsilon}_4}(w)$ is pictured in Figure 21 so that $rr(w) = 1\ 2\ 3\ 4\ 3\ 4\ 1\ 2\ 2$.

Since our insertion algorithm for basement $\bar{\epsilon}_n$ is just a twisted version of the RSK row insertion algorithm, it is easy to see that $rr(w) \to E^{\bar{\epsilon}_n} = P^{\bar{\epsilon}_n}(w)$. But then we know by part (3) of Corollary 10 that for all $j \geq 1$, the elements in the j-th row of $P^{\epsilon_n}(rr(w)) = rr(w) \to E^{\epsilon_n}$ is equal to the set of elements in the j-th row of $P^{\epsilon_n}(w)$ since both sets are equal to the elements in the j-th row of $P^{\bar{\epsilon}_n}(w) = P^{\bar{\epsilon}_n}(rr(w))$. Thus

$$\rho(P^{\epsilon_n}(w)) = \rho(P^{\epsilon_n}(rr(w))) = P^{\bar{\epsilon}_n}(w) = P^{\bar{\epsilon}_n}(rr(w)).$$

Since there is a unique PBF F with basement ϵ_n such that $\rho(F) = P^{\bar{\epsilon}_n}(w)$, we can conclude that $P^{\epsilon_n}(w) =$

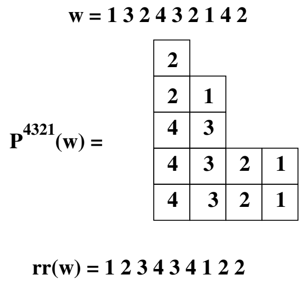


Figure 21: The reverse reading word $rr(w)$.

$P^{\epsilon_n}(rr(w))$. But then by part (1) of Corollary 10, it must be the case that $P^\sigma(w) = P^\sigma(rr(w))$ for all $\sigma \in S_n$. Thus we have the following Theorem.

Theorem 11. *(1) If $u, v \in \{1, \ldots, n\}^*$ and $P^{\bar{\epsilon}_n}(w) = P^{\bar{\epsilon}_n}(u)$, then $P^\sigma(u) = P^\sigma(w)$ for all $\sigma \in S_n$.*

(2) For any PBF *T with basement equal to $\bar{\epsilon}_n$ and any $\sigma \in S_n$, there is a unique* PBF *F^σ with basement σ such that $\rho_\sigma(F^\sigma) = T$.*

Theorem 11 says that we can construct $P^\sigma(w) = w \to E^\sigma$ by first constructing $P^{\bar{\epsilon}_n}(w) = w \to E^{\bar{\epsilon}_n}$ by our twisted version of RSK row insertion, then find $rr(w)$ which is the reverse reading word of $P^{\bar{\epsilon}_n}(w)$, and then compute $rr(w) \to E^\sigma$. However, it is easy to construct $rr(w) \to E^\sigma = \rho_\sigma^{-1}(P^{\bar{\epsilon}_n}(w))$. That is, suppose that $w = w_1 w_2 \ldots w_s$ is the strictly increasing word that results by reading the first row of $P^{\bar{\epsilon}_n}(w)$ in reverse reading order. Now consider inserting $w = w_1 w_2 \ldots w_s$ into E^σ. It is easy to see w_s will end up sitting on top of σ_i in the basement where i is the least j such that $\sigma_j \geq w_s$. Next consider the entry w_{s-1}. Before the insertion of w_s, w_{s-1} sat on top of σ_a in the basement where a is the least b such that $\sigma_b \geq w_{s-1}$. Now if a equals i, then w_s will bump w_{s-1} and w_{s-1} will move to σ_c where c is the least $d > i$ such that $\sigma_d \geq w_{s-1}$. Thus once w_s placed, w_{s-1} will be placed on top of σ_a in the basement where a is the least b such that $\sigma_b \geq w_{s-1}$ and w_s is not on top of σ_b. We continue this reasoning and show that $w = w_1 w_2 \ldots w_s \to E^\sigma$ can be constructed as inductively as follows:

First place w_s on top of σ_i in the basement where i is the least j such that $\sigma_j \geq w_s$. Once we have placed $w_s, \ldots, w_{r+1}$, place w_r on top of σ_u in the basement where u is the least v such that $\sigma_v \geq w_r$ and none of $w_s, \ldots, w_{r+1}$ are on top of σ_v.

It then should be clear that as we insert the second row, each element will sit on top of an element of in the first row according to the same procedure. That is, the inductive procedure to construct $\rho_\sigma^{-1}(P^{\bar{\epsilon}_n}(w))$ is essentially the same as the inductive procedure to construct ρ^{-1} except that the basement is σ instead of ϵ_n. We can record this formally as follows.

Procedure to construct $\rho_\sigma^{-1}(P^{\bar{\epsilon}_n}(w)) = rr(w) \to E^\sigma$.

Step 1. Let $w_1 \ldots w_s$ be the first row of $P^{\bar{\epsilon}_n}(w)$ in increasing order. First place w_s on top of σ_i in the basement where i is the least j such that $\sigma_j \geq w_s$. Then having placed $w_s, \ldots, w_{r+1}$, place w_r on top σ_u in the basement where u is the least v such that $\sigma_v \geq w_r$ and none of $w_s, \ldots, w_{r+1}$ are on top of σ_v.

Step $i > 0$. Inductively assume that the first i rows of $P^{\bar{\epsilon}_n}(w)$, $\{P_1, \ldots, P_{i-1}\}$ have been mapped to a PBF $F^{(i-1)}$ with basement σ in such a way that the elements in row j of $F^{(i-1)}$ are equal to P_j for $j = 1, \ldots, i-1$. Let $P_i = \{\alpha_1 > \alpha_2 > \cdots > \alpha_{s_i}\}$ be the i-th row of $P^{\bar{\epsilon}_n}(w)$. There exists an element greater than or equal to α_1 in row $i-1$ since there α_1 sits on top of some element in $P^{\bar{\epsilon}_n}(w)$. Place α_1 on top of the left-most such element in row $i-1$ of $F^{(i-1)}$. Next assume that we have placed $\alpha_1, \ldots, \alpha_{k-1}$. Then there are at least k elements of P_{i-1} that are greater than or equal to α_k since each of $\alpha_1, \ldots, \alpha_k$ sits on top of some element in row $i-1$ of $P^{\bar{\epsilon}_n}(w)$. Place α_k on top of the left-most element in row $i-1$ of $F^{(i-1)}$ which is greater than or equal to α_k which does not have one of $\alpha_1, \ldots, \alpha_{k-1}$ on top of it.

We then have the following theorem which shows that the insertion $w \to E^{\sigma}$ can be factored through the twisted RSK row insertion algorithm used to construct $w \to E^{\bar{\epsilon}_n}$.

Theorem 12. . *If $w \in \{1, \ldots, n\}^*$ and $\sigma \in S_n$, then $P^{\sigma}(w) = w \to E^{\sigma}$ equals $\rho_{\sigma}^{-1}(P^{\bar{\epsilon}_n}(w))$ where $P^{\bar{\epsilon}_n}(w) = w \to E^{\bar{\epsilon}_n}$.*

There are several important consequences of Theorem 12. First we will show that our insertion algorithm satisfies many of the properties that the usual RSK row insertion algorithm satisfies. Consider the usual Knuth equivalence relations for row insertion. Suppose that $u, v \in \{1, 2, \ldots\}^*$ and $x, y, z \in \{1, 2, \ldots\}$. The two types of Knuth relations are

(1) $uyxzv \backsim uyzxv$ if $x < y \leq z$ and
(2) $uxzyv \backsim uzxyv$ if $x \leq y < z$.

We say that two words $w, w' \in \{1, 2, \ldots\}^*$ are Knuth equivalent, $w \backsim w'$ if w can be transformed into w' by repeated use of (1) and (2). If $w \backsim w'$, then w and w' give us the same insertion tableau under row insertion. In our twisted version of row insertion the two types of Knuth relations become

$(1)^*$ $uyxzv \backsim^* uyzxv$ if $z \leq y < x$ and
$(2)^*$ $uxzyv \backsim^* uzxyv$ if $z < y \leq x$.

Then we say that two words $w, w' \in \{1, 2, \ldots, n\}^*$ are twisted Knuth equivalent, $w \backsim^* w'$, if w can be transformed to w' by repeated use of $(1)^*$ and $(2)^*$. Therefore if $w \backsim^* w'$, then $P^{\bar{\epsilon}_n}(w) = P^{\bar{\epsilon}_n}(w')$. Then Theorem 12 immediately implies the following.

Theorem 13. *Suppose that $w, w' \in \{1, 2, \ldots, n\}^*$ and $w \backsim^* w'$. Then for all $\sigma \in S_n$, $P^{\sigma}(w) = P^{\sigma}(w')$.*

It also follows from Theorem 12 that for every partition γ, the map ρ_{σ}^{-1} gives a one-to-one correspondence between the set of reverse row strict tableaux of shape γ whose entries are less than or equal to n and the set of PBFs with basement σ whose entries are less than or equal to n and whose shape is a rearrangement of γ. That is, we have the following theorem.

Theorem 14. *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a partition of n. Then*

$$s_{\gamma}(x_1, \ldots, x_n) = \sum_{\substack{\delta \\ \lambda(\delta)=\gamma}} \widehat{E}_{\delta}^{\sigma}(x_1, \ldots, x_n) \tag{5}$$

where the sum runs over all weak compositions $\delta = (\delta_1, \ldots, \delta_n)$ which are rearrangements of γ.

Note that in Theorem 14, the set of weak compositions that actually appear on the right hand side of (5) depends on σ. That is, we know by Corollary 10 that if $\sigma_i > \sigma_j$, then in a PBF F^{σ} with basement $\sigma = \sigma_1 \ldots \sigma_n$, the height of column i must be greater than or equal to the height of column j in F^{σ}. For example, consider $s_{(2,1,0)}(x_1, x_2, x_3)$. In Figure 4 we have listed the eight PBFs with basement $\bar{\epsilon}_3 = 3\ 2\ 1$ over the alphabet $\{1, 2, 3\}$. Below each of these PBFs G, we have pictured $\rho_{123}^{-1}(G)$, $\rho_{132}^{-1}(G)$ and $\rho_{312}^{-1}(G)$. One can see that

$$\begin{aligned} s_{(2,1,0)}(x_1, x_2, x_3) &= \widehat{E}_{(2,1,0)}^{123}(x_1, x_2, x_3) + \widehat{E}_{(2,0,1)}^{123}(x_1, x_2, x_3) + \widehat{E}_{(1,2,0)}^{123}(x_1, x_2, x_3) + \\ &\quad \widehat{E}_{(1,0,2)}^{123}(x_1, x_2, x_3) + \widehat{E}_{(0,2,1)}^{123}(x_1, x_2, x_3) + \widehat{E}_{(0,1,2)}^{123}(x_1, x_2, x_3), \end{aligned}$$

$$s_{(2,1,0)}(x_1, x_2, x_3) = \widehat{E}_{(2,1,0)}^{132}(x_1, x_2, x_3) + \widehat{E}_{(1,2,0)}^{132}(x_1, x_2, x_3) + \widehat{E}_{(0,2,1)}^{132}(x_1, x_2, x_3),$$

and

$$s_{(2,1,0)}(x_1, x_2, x_3) = \widehat{E}_{(2,1,0)}^{312}(x_1, x_2, x_3) + \widehat{E}_{(2,0,1)}^{312}(x_1, x_2, x_3).$$

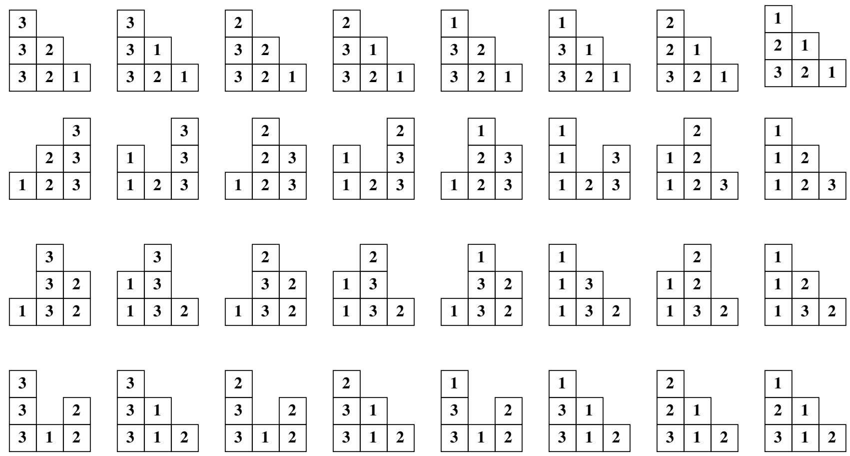


Figure 22: PBFs corresponding to $s_{(2,1,0)}(x_1, x_2, x_3)$.

5 Pieri rules

The homogeneous symmetric function $h_k(x_1, \ldots, x_n)$ and the elementary symmetric function $e_k(x_1, \ldots, x_n)$ are defined by

$$\begin{aligned} h_k(x_1, \ldots x_n) &= \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} \text{ and} \\ e_k(x_1, \ldots x_n) &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}. \end{aligned}$$

The Pieri rules for Schur functions state that

$$h_k(x_1, \ldots x_n) s_\mu(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} s_\lambda(x_1, \ldots, x_n) \tag{6}$$

where the sum runs over all λ such that λ/μ does not contain two elements in the same column and that

$$e_k(x_1, \ldots x_n) s_\mu(x_1, \ldots, x_n) = \sum_{\mu \subseteq \lambda} s_\lambda(x_1, \ldots, x_n) \tag{7}$$

where the sum runs over all λ such that λ/μ does not contain two elements in the same row.

The main goal of this section is to prove an analogue of the Pieri rules for the products

$$\begin{aligned} &h_k(x_1, \ldots x_n) \widehat{E}_\gamma^\sigma(x_1, \ldots, x_n) \text{ and} \\ &e_k(x_1, \ldots x_n) \widehat{E}_\gamma^\sigma(x_1, \ldots, x_n). \end{aligned}$$

We start with a simple lemma about the effect of inserting two letters into a PBF.

Lemma 15. *Suppose that F^σ is a* PBF, *$G^\sigma = k \to F^\sigma$ and $H^\sigma = k' \to G^\sigma$. Suppose F^σ is of shape α, G^σ is of shape β, H^σ is of shape γ, T is the cell in $dg'(\beta) - dg'(\alpha)$, and T' is the cell in $dg'(\gamma) - dg'(\beta)$. Then*

1. *if $k \geq k'$, then T is strictly below T' and*
2. *if $k < k'$, then T appears before T' in reading order.*

Proof. There is no loss in generality in assuming that $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ where $n \geq max(k, k')$. Assume $\bar{F}^\sigma$ is the diagram that results by adding 0's on top of the cells of F^σ as in the definition of the insertion

$k \to F^\sigma$. Let $c_1, c_2, \ldots$ be the cells in reading order that are in $\bar{F}^\sigma$ but not in the basement. We will prove this result by induction on the number of cells p in the list $c_1, c_2, \ldots$.

First suppose that $p = 0$ so that F^σ just consists of the basement permutation σ. Then k will be inserted in cell $(i, 1i)$ where i is the least j such that $k \leq \sigma_j$. Now if $k' \leq k$, then it is easy to see that k' will be inserted on top of k in the insertion $k' \to G^\sigma$ so that T will be strictly below T'.

If $k' > k$, then suppose that k is in position $(i, 1)$ in G^σ. Then it is clear that k' cannot be placed in any of the positions $(j, 1)$ with $j < i$ since k could not be placed in any of those positions. Hence k' either bumps k or is placed in the first row in some cell to the right of k. In either case, T preceeds T' in reading order.

Now if $p > 0$, there are two cases.

Case 1. k is placed in cell c_i which is either equal to c_1 or in the same row as c_1.

In either case, c_i is a cell on top of a column in F^σ. Let $\overline{c_i}$ be the cell immediately above c_i. Then the cell $\overline{c_i}$ will be the first cell in reading order in $\bar{G}^\sigma$. If $k' \leq k$, then k' will be placed in $\overline{c_i}$ so that $c_i = T$ and $\overline{c_i} = T'$. Thus T will occur below T'.

If $k < k'$, then k' cannot be placed in $\overline{c_i}$. Thus either k' bumps k in cell c_i or it is inserted into cells of $\bar{F}^\sigma$ after c_i. In either case, T' follows $c_i = T$ in reading order.

Case 2. k is placed in cell a c_i which is not in the same row as c_1.

Let c_j be the first cell in our list which is not in the same row as cell c_1. If k' is not placed in any of the cells $c_1, \ldots c_{j-1}$, then we are inserting k followed by k' into the sequence $\bar{F}^\sigma(c_j), \bar{F}^\sigma(c_j + 1), \ldots$ so the result follows by induction. However, the only way that k' can be placed in a cell c_i in the same row as c_1 is if $k' < k$ in which case $T' = c_i$. In that case, T lies in a row below the row of c_1 so that T lies strictly below T'. □

Theorem 16. *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a weak composition of p and $\sigma \in S_n$. Then*

$$h_k(x_1, \ldots x_n) \widehat{E}^\sigma_\gamma(x_1, \ldots, x_n) = \sum_\alpha \widehat{E}^\sigma_\alpha(x_1, \ldots, x_n), \tag{8}$$

where the sum runs over all weak compositions $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $p + k$ such that (i) $dg'(\gamma) \subseteq dg'(\alpha)$, (ii) $dg'(\alpha)/dg'(\gamma)$ has no two elements in the same row, and (iii) if $(x, y) \in dg'(\alpha)/dg'(\gamma)$, then (x, y) must be at the top of column the rightmost column x of height y in $dg'(\alpha)$, i.e. $\alpha_x = y$, and if $x < z$, then $\alpha_x \neq \alpha_z$. (Condition (iii) says that (x, y) must always be the right-most cell at the top of a set of columns that have the same height in α.) *Similarly,*

$$e_k(x_1, \ldots x_n) \widehat{E}^\sigma_\gamma(x_1, \ldots, x_n) = \sum_\beta \widehat{E}^\sigma_\beta(x_1, \ldots, x_n), \tag{9}$$

where the sum runs over all weak compositions $\beta = (\beta_1, \ldots, \beta_n)$ such that (a) $dg'(\gamma) \subseteq dg'(\beta)$, (b) $dg'(\beta)/dg'(\gamma)$ has no two elements in the same column, and (c) if $(x, y) \in dg'(\beta)/dg'(\gamma)$, then (x, y) must be at the top of column x in $dg(\beta)$, i.e. $\beta_x = y$, and if $x < n$ and $\beta_x = \beta_{x+1}$, then $(x + 1, y) \in dg'(\beta)/dg'(\gamma)$.
(Condition (c) says that if we remove elements from the tops of a sequence of columns in β to get γ, then that set of elements is a set of elements reading from right to left.)

Proof. The left hand side of (8) can be interpreted as the weight of the set of pairs (w, F^σ) where $w = w_1 \ldots w_k$ and $n \geq w_1 \geq \cdots \geq w_k \geq 1$, F^σ is a PBF of shape γ with basement σ, and the weight $W(w, F^\sigma)$ of the pair (w, F^σ) is equal to $W(F^\sigma) \prod_{i=1}^k x_{w_i}$. The right hand side of (8) can be interpreted as the sum of the weights of all PBFs G^σ with basement σ such G^σ has shape $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some α which is a weak composition of $p + k$ such that $dg'(\alpha)$ satisfies conditions (i), (ii), and (iii).

Now consider the map Θ which takes such a pair (w, F^σ) to

$$\Theta(w, F^\sigma) = w \to F^\sigma = G^\sigma.$$

Let $G^\sigma_i = w_1 \ldots w_i \to F^\sigma$ for $i = 1, \ldots, k$ and let $G^\sigma_0 = F^\sigma$. Then let c_i be the cell in $dg'(G^\sigma_i)/dg(G^\sigma_{i-1})$ for $i = 1, \ldots, k$. It follows that G^σ must be of shape $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α is a weak composition of $p + k$.

By Lemma 15, we know that c_{i+1} must be strictly above c_i for $i = 1, \ldots, k-1$. Also, by Proposition 7, we know that c_i must be at the top of some column of $dg'(\alpha)$ and that no column to the right in $dg'(\alpha)$ can have the same height as the height of the column that contains c_i in α. It follows that G^σ is a PBF of some shape $\alpha = (\alpha_1, \ldots, \alpha_n)$ which satisfies conditions (i), (ii), and (iii) of the Theorem. Moreover, it is clear that $W(G^\sigma) = W(w, F^\sigma)$. Since our insertion procedure can be reversed, it is easy to see that Θ is one-to-one.

To see that Θ is a bijection between the pairs (w, F^σ) contributing to the left hand side of (8) and the PBFs G^σ contributing to the right hand side (8), we must show that for each G^σ contributing to the right hand side (8), there is a pair (w, F^σ) contributing to the left hand side (8) such that $w \to F^\sigma = G^\sigma$. Suppose that G^σ is a PBF with basement σ such G^σ has shape $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some α that meets conditions (i)-(iii). Let $c_k, \ldots, c_1$ be the cells of $dg'(\alpha)/dg'(\gamma)$ read from top to bottom. Because c_k is at the right-most top of a sequence of columns in α of the same height, it follows from our remarks following Proposition 7 that we can reverse the insertion procedure starting at cell c_k. Thus we can first reverse the insertion process for the element in cell c_k in G^σ to produce a PBF F^σ_{k-1} with basement σ and shape α with c_k removed and a letter w_k such that $w_k \to F^\sigma_{k-1} = G^\sigma$. Then we can reverse our insertion process for the element in cell c_{k-1} of F^σ_{k-1} to produce a PBF F^σ_{k-2} with basement σ and shape α with c_k and c_{k-1} removed and a letter w_{k-1} such that $w_{k-1}w_k \to F^\sigma_{k-2} = G^\sigma$. Continuing on in this manner we can produce a sequence of PBF's $F^\sigma_0, \ldots, F^\sigma_{k-1}$ with basement σ and a word $w = w_1 \ldots w_k$ such that $w_i \ldots w_k \to F^\sigma_{i-1} = G^\sigma$ and the shape of F^σ_{i-1} equals α with the cells $c_i, c_{i+1}, \ldots, c_k$ removed. Thus F^σ_0 will be a PBF with basement σ and shape α such that $w \to F^\sigma_0 = G^\sigma$. The only thing that we have to prove is that $w_1 \geq \cdots \geq w_k$. But it cannot be that $w_i < w_{i+1}$ for some i because Lemma 15 would imply that c_i appears before c_{i+1} in reading order which it does not. Thus Θ is a bijection which proves that (8) holds.

The left hand side of (9) can be interpreted as the weight of the set of pairs (u, H^σ) where $u = u_1 \ldots u_k$ and $1 \leq u_1 < \cdots < u_k \leq n$, H^σ is a PBF of shape γ with basement σ, and the weight $W(u, H^\sigma)$ of the pair (u, H^σ) is equal to $W(H^\sigma)\prod_{i=1}^k x_{u_i}$. The right hand side of (9) can be interpreted as the sum of the weights of all PBFs K^σ with basement σ such that K^σ has shape $\beta = (\beta_1, \ldots, \beta_n)$ for some β which satisfies conditions (a)-(c) of the Theorem.

Again consider the map Θ which takes such a pair (u, H^σ) to

$$\Theta(u, H^\sigma) = u \to H^\sigma = K^\sigma.$$

Let $K^\sigma_i = u_1 \ldots u_i \to H^\sigma$ for $i = 1, \ldots, k$ and let $K^\sigma_0 = H^\sigma$. Then let c_i be the cell in $dg'(K^\sigma_i)/dg'(K^\sigma_{i-1})$ for $i = 1, \ldots, k$. By Lemma 15 we know that c_i must appear before c_{i+1} in reading order for $i = 1, \ldots, k-1$. Since c_{i+1} sits on the outside of the shape of K^σ_i, it cannot be that c_i and c_{i+1} are in the same column because then c_{i+1} would have to sit on top of c_i and c_i would not appear before c_{i+1} in reading order. In fact, it cannot sit on top of any of the cells $c_1, \ldots, c_i$ for the same reason. It follows that K^σ is a PBF of some shape $\beta = (\beta_1, \ldots, \beta_n)$ such that the $dg'(\gamma) \subseteq dg'(\beta)$ and $dg'(\beta)/dg'(\gamma)$ has no two elements in the same column and β is a weak composition of $p + k$. It follows from Proposition 7 that for a maximal set S of columns of the same height in β, the only elements in those columns which could be among the set of c_i's is a consecutive set of elements at the tops of the those columns reading from right to left. It follows that β must also satisfy condition (c) of the Theorem. Moreover, it is clear that $W(G^\sigma) = W(w, F^\sigma)$. Since our insertion procedure can be reversed, it is easy to see that Θ is one-to-one.

To see that Θ is a bijection between the pairs (u, H^σ) contributing to the left hand side of (9) and the PBFs K^σ contributing to the right hand side (9), we must show that for each K^σ contributing to the right hand side of (9), there is a pair (u, H^σ) contributing to the left hand side of (9) such that $u \to H^\sigma = K^\sigma$. So suppose that K^σ is a PBF with basement σ such K^σ has shape $\beta = (\beta_1, \ldots, \beta_n)$ which is a weak composition of $p + k$ which satisfies conditions (a)-(c).

Let $c_k, \ldots, c_1$ be the cells of $dg'(\beta)/dg'(\gamma)$ read in reverse reading order. Since $dg'(\beta)/dg'(\gamma)$ has no two elements in the same column and β satisfies condition (c), it is not difficult to see that c_k must be on the outside of shape β. Moreover, c_k is the right-most cell at the top of a set of columns of the same height in the shape of β so that we can reverse our insertion process starting at cell c_k. Similarly, c_{k-1} is on the outside of β with c_k removed, and, in general, c_i must be on the outside of β with $c_k, \ldots, c_{i+1}$ removed. Moreover, since β satisfies condition (c), c_i is always the right-most cell at the top of a set of columns of the same height in the shape of β with $c_k, \ldots, c_{i+1}$ removed. This means that we can reverse our insertion process starting with cell c_i after we have reversed the insertion process starting at cells $c_k, \ldots, c_{i+1}$. Then we first reverse the insertion process for the element in cell c_k in K^σ to produce a PBF H^σ_{k-1} with basement σ and shape β with c_k removed and a letter u_k such that $u_k \to H^\sigma_{k-1} = K^\sigma$. Then we can reverse our insertion process for the element in cell c_{k-1} of H^σ_{k-1} to produce a PBF H^σ_{k-2} with basement σ and shape β with c_k and c_{k-1} removed and a letter u_{k-1} such that $u_{k-1}u_k \to H^\sigma_{k-2} = K^\sigma$. Continuing in this manner we can produce a sequence of PBF's $H^\sigma_0, \ldots, H^\sigma_{k-1}$ with basement σ and a word $u = u_1 \ldots u_k$ such that $w_i \ldots w_k \to H^\sigma_{i-1} = G^\sigma$ and the shape of H^σ_{i-1} equals β with the cells $c_i, c_{i+1}, \ldots, c_k$ removed. Thus H^σ_0 will be a PBF with basement σ and shape β such that $u \to H^\sigma_0 = K^\sigma$. The only thing that we have to prove is that $u_1 < \cdots < u_k$. But it cannot be that $u_i \geq u_{i+1}$ for some i because Lemma 15 would force c_{i+1} to appear in a row which is strictly above the row in which c_i appears which would mean that c_{i+1} does not follow c_i in reading order. Thus Θ is a bijection which proves that (9) holds.

□

6 A permuted basements analogue of the Robinson-Schensted-Knuth algorithm

We are now ready to state an analogue of the Robinson-Schensted-Knuth Algorithm for PBF's.

Let $A = (a_{i,j})$ be an arbitrary $n \times n$-matrix with nonnegative integer entries and let $\sigma = \sigma_1 \ldots \sigma_n \in S_n$. For each pair i, j such that $a_{i,j} > 0$, create a sequence of $a_{i,j}$ biletters $\begin{matrix} i \\ j \end{matrix}$. Let w_A be the unique two-line array consisting of such biletters so that the top letters are weakly increasing and for all pairs with the same top letter the bottoms letters are weakly increasing. For example, if

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then

$$w_A = \begin{matrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 1 & 2 & 4 & 1 & 2 & 4 & 4 & 3 & 3 & 4 & 1 & 2 & 5 \end{matrix}.$$

Let u_A be the word consisting of the top row of w_A and v_A be the word consisting of the bottom row of w_A. Let $P^\sigma_0 = Q^\sigma_0 = E^\sigma$ be empty PBFs with basement σ. We say that P^σ_0 is the initial insertion PBF and Q^σ_0 is the initial recording PBF relative to σ.

Now suppose that $u_A = i_1 \ldots i_t$ and $v_A = j_1 \ldots j_t$. Then insert the biletters of w_A into the insertion and recording PBFs using the following inductive procedure. Assume that the last k biletters of w_A have already been inserted and the resulting pair of PBFs is $(\mathrm{P_k}^\sigma, \mathrm{Q_k}^\sigma)$. Map the entry j_{t-k} into $\mathrm{P_k}^\sigma$ according to the procedure $j_{t-k} \to \mathrm{P_k}^\sigma$. The resulting filling is P^σ_{k+1}. Record the position of the new entry by placing the entry i_{t-k} into the leftmost empty cell in this row of $\mathrm{Q_k}^\sigma$ which lies immediately above a cell greater than or equal to i_{t-k}. The resulting filling is Q^σ_{k+1}. Repeat this procedure until all of the biletters from w_A have been inserted. The resulting pair $(\mathrm{P_n}^\sigma, \mathrm{Q_n}^\sigma) := (\mathrm{P}^\sigma, \mathrm{Q}^\sigma)$ is denoted by $\Psi_\sigma(A)$. For example, if $\sigma = 1\ 4\ 2\ 5\ 3$ and A is the matrix given above, then $\Psi_\sigma(A) = (P^\sigma, Q^\sigma)$ is pictured in Figure 23.

Next consider the special case where $\sigma = \bar{\epsilon}_n$. Note that $P^{\bar{\epsilon}_n} = j_t j_{t-1} \ldots j_1 \to E^{\bar{\epsilon}_n}$ is constructed by a twisted version of the usual RSK row insertion algorithm. In that case, the recording PBF $Q^{\bar{\epsilon}_n}$ is constructed in the same way that the usual RSK recording tableau is constructed except that we are constructing tableaux such that columns are weakly decreasing reading from bottom to top and the rows are strictly decreasing reading from left to right. Thus $\Psi_{\bar{\epsilon}_n}$ is just a twisted version of the usual RSK correspondence between $\mathbb{N}$-valued $n \times n$-matrices and pairs of column strict tableaux of the same shape. In particular, we know that if A is an $\mathbb{N}$-valued $n \times n$-matrix and A^T is its transpose, then $\Psi_{\bar{\epsilon}_n}(A) = (P^{\bar{\epsilon}_n}, Q^{\bar{\epsilon}_n})$ if and only if $\Psi_{\bar{\epsilon}_n}(A^T) = (Q^{\bar{\epsilon}_n}, P^{\bar{\epsilon}_n})$.

Theorem 17. *Let $\sigma = \sigma_1 \ldots \sigma_n \in S_n$. The map Ψ_σ is a bijection between $\mathbb{N}$-valued $n \times n$ matrices and pairs $(\mathrm{P}^\sigma, \mathrm{Q}^\sigma)$ of* PBF*'s of shapes (α, β) and basement σ such that $\lambda(\alpha) = \lambda(\beta)$, $\alpha_i \geq \alpha_j$, and $\beta_i \geq \beta_j$ whenever $i < j$ and $\sigma_i > \sigma_j$.*

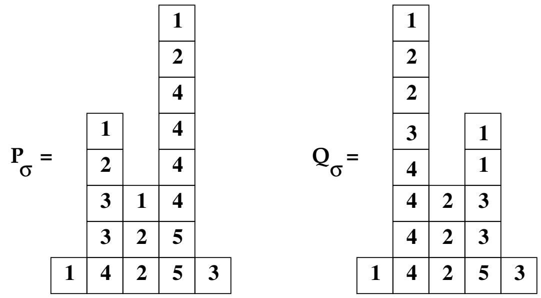
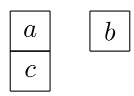


Figure 23: $\Psi_\sigma(w_A) = (P^\sigma, Q^\sigma)$.

Proof. Suppose that A is an $\mathbb{N}$-valued $n \times n$ matrix and $\Psi_\sigma(A) = (\mathrm{P}^\sigma, \mathrm{Q}^\sigma)$. The filling P^σ is a PBF by Lemma 6. The shape α of P^σ satisfies $\alpha_i \geq \alpha_j$ for all inversions $i < j$ of σ by Corollary 10. It is also easy to see that our definition of Ψ_σ ensures that $\lambda(\alpha) = \lambda(\beta)$.

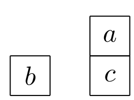
We must prove that the filling Q^σ is a PBF. The columns of Q^σ are weakly decreasing from bottom to top by construction. For any given i, the bottom elements of biletters whose top elements are i are inserted in weakly decreasing order. It then follows from Lemma 15 that i cannot occur twice in the same row in Q^σ.

To see that every triple is an inversion triple, consider first a type A triple consisting of the cells $a = (x_1, y_1)$, $b = (x_2, y_1)$, and $c = (x_1, y_1 - 1)$ where $x_1 < x_2$ as depicted below.



This triple would be a type A coinversion triple only if $Q^\sigma(a) \leq Q^\sigma(b) \leq Q^\sigma(c)$. Since we can not have two equal elements in Q^σ in the same row, it must be that $Q^\sigma(a) < Q^\sigma(b) \leq Q^\sigma(c)$. There are now two cases. First if $Q^\sigma(b) < Q^\sigma(c)$, then under the Ψ_σ map, $Q^\sigma(c)$ was placed first in Q^σ, then $Q^\sigma(b)$ was placed, and then $Q^\sigma(a)$ was placed. But this means at the time $Q^\sigma(b)$ was placed, it could have been placed on top of $Q^\sigma(c)$ which is a contradiction since the Ψ_σ map requires that $Q^\sigma(b)$ be placed in the left-most possible position subject to the requirement that the columns are weakly decreasing. The second case is when $Q^\sigma(b) = Q^\sigma(c)$. In that case, Lemma 15 ensures that the cells created by the insertion of the bottoms of biletters whose tops equal $Q^\sigma(b)$ are created from bottom to top. This means that the biletter which created cell c in Q^σ must have been processed before the biletter which created cell b. But this means that under the Ψ_σ map, $Q^\sigma(c)$ was placed first in Q^σ, then $Q^\sigma(b)$ was placed, and then $Q^\sigma(a)$ was placed which we have already determined is impossible. Thus there are no type A coinversion triples in Q^σ.

Now suppose that there exists $a = (x_2, y)$, $b = (x_1, y-1)$, and $c = (x_2, y-1)$ (where $x_1 \leq x_2$) which form a type B coinversion triple in Q^σ as depicted below.



We know that $Q^\sigma(b) \neq Q^\sigma(c)$ since we cannot have two equal elements in the same row in Q^σ. Thus we must have $Q^\sigma(a) \leq Q^\sigma(b) < Q^\sigma(c)$. Now if $Q^\sigma(a) < Q^\sigma(b)$, then under the Ψ_σ map, $Q^\sigma(c)$ was placed first in Q^σ, then $Q^\sigma(b)$ was placed, and then $Q^\sigma(a)$ was placed. However, if $Q^\sigma(a) = Q^\sigma(b)$, then Lemma 15 ensures that the cells created by the insertion of the bottoms of biletters whose tops equal $Q^\sigma(b)$ are created from bottom to top. This means that the biletter which created cell b in Q^σ must have been processed before the biletter which created cell a. Thus in either case, under the Ψ_σ map, $Q^\sigma(c)$ was placed first in Q^σ, then $Q^\sigma(b)$ was placed, and then $Q^\sigma(a)$ was placed. The only reason that $Q^\sigma(a)$ was not placed on top of $Q^\sigma(b)$ is that there must have already existed an element e which was on top of $Q^\sigma(b)$ at the time $Q^\sigma(a)$ was placed. This means that $Q^\sigma(a) \leq e$ since the cells in Q^σ are created by adding elements in weakly decreasing order. However since we can not have two equal elements in the same row, we must have that $Q^\sigma(a) < e$. Thus we know $\mathrm{Q}^\sigma(x_1, y) > \mathrm{Q}^\sigma(x_2, y)$. But this means that if we added an element z in cell $(x_2, y+1)$ which sits on top of $Q^\sigma(a)$, then the only reason that z was not placed on top of $e = \mathrm{Q}^\sigma(x_1, y)$ is that there must have already been an element in $\mathrm{Q}^\sigma(x_1, y+1)$ at the time we added z. But then we can argue as above that it must be the case that $\mathrm{Q}^\sigma(x_1, y+1) > \mathrm{Q}^\sigma(x_2, y+1)$. Then we can repeat the argument for row $y+2$ so that if $(x_2, y+2)$ is a cell in Q^σ, then $(x_1, y+2)$ must have already been filled at the time we added an element to $(x_2, y+2)$ and that $\mathrm{Q}^\sigma(x_1, y+2) > \mathrm{Q}^\sigma(x_2, y+2)$. Continuing on in this way, we conclude that the height of column x_1 in Q^σ is greater than or equal to the height of column x_2 in Q^σ. But that is a contradiction, since if $\{a, b, c\}$ is a type B triple, the height of column x_1 in Q^σ must be less than the height of column x_2 in Q^σ. Thus there can be no type B coinversion triples in Q^σ.

Note that our argument above did not really use any properties of $Q^\sigma(c)$, but only relied on the fact that $Q^\sigma(a) \le Q^\sigma(b)$. That is, we proved that if $x_1 < x_2$ and $\mathrm{Q}^\sigma(x_1, y-1) \ge \mathrm{Q}^\sigma(x_2, y)$, then the height of column x_1 in Q^σ must be greater that or equal to the height of column x_2 in Q^σ. But this means that if the height of column x_1 in Q^σ is less than the height of column x_2 in Q^σ, then $\mathrm{Q}^\sigma(x_1, y-1) < \mathrm{Q}^\sigma(x_2, y)$, which is precisely the B-increasing condition. Thus Q^σ is a PBF.

Next consider the shape β of Q^σ. We must prove that $\beta_i \ge \beta_j$ for all inversions $i < j$ of σ. Consider the shape $\beta^{(1)}$ of Q^σ after we have placed j_n into Q^σ. Since j_n is placed on top of the leftmost entry $\sigma(k)$ such that $\sigma(k) \ge j_n$, the first $k-1$ entries of σ are less than $\sigma(k)$ and hence the claim is satisfied after the initial insertion.

Assume that the claim is satisfied after the insertion of each of the last $k-1$ letters of w_A and consider the placement of the entry j_{n-k} in Q^σ. Let s be the index of the column into which j_{n-k} is placed. Let t be an integer less than s such that $\sigma(t) > \sigma(s)$. Then column t is weakly taller than column s before the placement of j_{n-k} by assumption. If column t is strictly taller, then the placement of j_{n-k} on top of column s will not alter the relative orders of the columns. If the heights of columns t and s are equal, then the highest entry in column t was inserted before the highest entry in column s, for otherwise the columns would violate the condition immediately after the highest entry of column s was inserted. But then j_{n-k} would be inserted on top of column t, a contradiction. Therefore the shape β of Q^σ satisfies the condition that $\beta_i \ge \beta_j$ for all pairs (i, j) satisfying $i < j$ and $\sigma_i > \sigma_j$.

Thus we know that Ψ_σ maps any $n \times n$ matrix A to a pair of PBFs (P^σ, Q^σ). Now suppose that $u_A = i_1 \dots i_t$ and $v_A = j_1 \dots j_t$ and $\sigma = \sigma_1 \dots \sigma_n$ where $\sigma_i < \sigma_{i+1}$. We would like to determine the relationship between Q^σ and $Q^{s_i\sigma}$. We established in Corollary 9 that as we consider the sequence of insertions

$$\begin{array}{cc} j_t \to E^\sigma & j_t \to E^{s_i\sigma} \\ j_t j_{t-1} \to E^\sigma & j_t j_{t-1} \to E^{s_i\sigma} \\ \vdots & \vdots \\ j_t \dots j_1 \to E^\sigma & j_t \dots j_1 \to E^{s_i\sigma}, \end{array}$$

the new cells that we created by the insertions at each stage were in the same row of E^σ as in $E^{s_i\sigma}$. This implies that for all j, the elements in row j of Q^σ and $Q^{s_i\sigma}$ are the same. But then it is easy to prove by induction on the number of inversions of σ that for all j, the elements in row j of Q^σ and $Q^{\bar{\epsilon}_n}$ are the same. That is, $\rho(Q^\sigma) = Q^{\bar{\epsilon}_n}$. Since there is a unique PBF Q with basement σ such that for all j, the elements in row j of Q and $Q^{\bar{\epsilon}_n}$ are the same, it follows that $Q^\sigma = \rho_\sigma^{-1}(Q^{\bar{\epsilon}_n})$ for all σ. Since $P^\sigma = j_t j_{t-1} \dots j_1 \to E^\sigma$ for all σ, we know by the results of Section 3 that $P^\sigma = \rho_\sigma^{-1}(P^{\bar{\epsilon}_n})$ for all σ. Thus it follows that for any $\mathbb{N}$-valued $n \times n$ matrix A,

$$\Psi_\sigma(A) = (P^\sigma, Q^\sigma) = (\rho_\sigma^{-1}(P^{\bar{\epsilon}_n}), \rho_\sigma^{-1}(Q^{\bar{\epsilon}_n})).$$

Since $\Psi_{\bar{\epsilon}_n}$ and ρ_σ^{-1} are bijections, it follows that Ψ_σ is also a bijection between $\mathbb{N}$-valued $n \times n$ matrices A and pairs (P, Q) of PBFs with basement σ.

We note that another way to define the inverse of Ψ_σ is given by choosing the first occurrence (in reading order) of the smallest value in Q^σ, removing it from Q^σ, and labeling this entry j_1. Then choose the rightmost entry in this row of P^σ which sits at the top of its column and apply the inverse of the insertion procedure to remove this cell from P^σ. The resulting entry is then i_1. Repeat this procedure to obtain the array w_A.

□

Note that our proof of Theorem 17 allows us to prove the following corollary which says that for any $\sigma \in S_n$, the map Ψ_σ can be factored through our twisted version of the RSK correspondence.

Corollary 18. *For any* $\mathbb{N}$*-valued* $n \times n$ *matrix* A*,*

$$\Psi_\sigma(A) = (P^\sigma, Q^\sigma) = (\rho_\sigma^{-1}(P^{\bar{\epsilon}_n}), \rho_\sigma^{-1}(Q^{\bar{\epsilon}_n})) \tag{10}$$

where the map

$$\Psi_{\bar{\epsilon}_n}(A) = (P^{\bar{\epsilon}_n}, Q^{\bar{\epsilon}_n}) \tag{11}$$

is a twisted version of the usual RSK correspondence.

Corollary 18 allows us to prove that our permuted basement version of the RSK correspondence Ψ_σ satisfies many of the properties that are satisfied by the RSK correspondence. For example, we have the following theorem.

Theorem 19. *Suppose that* A *is an* $\mathbb{N}$*-valued* $n \times n$ *matrix and* A^T *is its transpose. Then for all* $\sigma \in S_n$*,*

$$\Psi_\sigma(A) = (P^\sigma, Q^\sigma) \iff \Psi_\sigma(A^T) = (Q^\sigma, P^\sigma). \tag{12}$$

Proof. By the usual properties of the RSK correspondence, we know that

$$\Psi_{\bar{\epsilon}_n}(A) = (P^{\bar{\epsilon}_n}, Q^{\bar{\epsilon}_n}) \iff \Psi_{\bar{\epsilon}_n}(A^T) = (Q^{\bar{\epsilon}_n}, P^{\bar{\epsilon}_n}). \tag{13}$$

Then (12) follows immediately from (13) and (10). □

6.1 Standardization

Let $w = w_1 \dots w_n \in \{1, \dots, n\}^*$ be a word and let $P^\sigma(w) = w_1 \dots w_n \to E^\sigma$. One can standardize w in the usual manner. That is, if w has i_j j's for $j = 1, \dots n$, then the *standardization of* w, $st(w)$, is the permutation that results by replacing the 1's in w by $1, \dots, i_1$, reading from right to left, then replacing the 2's in w by $i_1 + 1, \dots, i_1 + i_2$, reading from right to left, etc.. If $st(w_1 \dots w_n) = s_1 \dots s_n$, then we define the standardization of $P^\sigma(w)$ by letting $st(P^\sigma(w)) = s_1 \dots s_n \to E^\sigma$.

In the special case where $\sigma = \bar{\epsilon}_n$, there are two different ways to find $st(P^\sigma(w))$. That is, we can compute $st(P^{\bar{\epsilon}_n}) = st(w) \to E^{\bar{\epsilon}_n}$ directly or we can compute $P^{\bar{\epsilon}_n} = w \to E^{\bar{\epsilon}_n}$ and then standardize the reverse row strict tableau $P^{\bar{\epsilon}_n}$. Here, for any reverse row strict tableau T, $st(F)$ is the standard reverse row strict tableau obtained by replacing the 1's in T by $1, \dots, i_1$ in order from top to bottom, then replacing the 2's in T by $i_1 + 1, \dots, i_1 + i_2$, reading from top to bottom, etc.. This follows from the fact that the usual standardization operations for words and column strict tableaux commute with RSK row insertion; see [10]. Thus our standardization operations for words and reverse row strict tableaux commute with our twisted version of RSK row insertion. That is, suppose $w = w_1 \dots w_n \in \{1, \dots, n\}^*$ and $st(w) = s_1 \dots s_n$. Then $w \to E^{\bar{\epsilon}_n} = T$ if and only if $s_1 \dots s_n \to E^{\bar{\epsilon}_n} = st(T)$. Because our insertion algorithm where the basement permutation is $\bar{\epsilon}_n$ can be factored through our twisted version of RSK row insertion, the same thing happens when the basement is σ. That is,

$$\begin{aligned} st(P^\sigma(w)) &= s_1 \dots s_n \to E^\sigma \\ &= \rho_\sigma^{-1}(P^{\bar{\epsilon}_n}(st(w))) \\ &= \rho_\sigma^{-1}(st(P^{\bar{\epsilon}_n}(w))). \end{aligned}$$

We can summarize the above discussion in the following two propositions.

Proposition 20. *Let* $w = w_1 \dots w_n \in \{1, \dots, n\}^*$ *and* $st(w) = s_1 \dots s_n$*. If* $P^\sigma(w) = w_1 \dots w_n \to E^\sigma$ *is of shape* γ *where* $|\gamma| = n$*, then the* PBF $st(P^\sigma(w)) = s_1 \dots s_n \to E^\sigma$ *is a* PBF *whose shape is a rearrangement of* γ*.*

Proof. We have proved above that $st(P^\sigma(w)) = \rho_\sigma^{-1}(st(P^{\bar{\epsilon}_n}(w)))$. Since the shape of $st(P^{\bar{\epsilon}_n}(w))$ is $\lambda(\gamma)$, we know that $\rho_\sigma^{-1}(st(P^{\bar{\epsilon}_n}(w)))$ is a rearrangement of γ. □

Proposition 21. *The standardization of words and* PBF*s commutes with our insertion algorithm relative to the basement σ in the sense that for any $w = w_1 \ldots w_n \in \{1, \ldots, n\}^*$, we have the following commutative diagram.*

$$\begin{array}{ccc} P^\sigma(w) & \longrightarrow & P^\sigma(st(w)) = st(P^\sigma(w)) \\ \rho_\sigma \Big\downarrow & & \Big\uparrow \rho_\sigma^{-1} \\ T & \longrightarrow & st(T) \end{array}$$

A specific example of this process for $w = 4\ 3\ 1\ 3\ 2\ 3\ 4\ 1$ is pictured in Figure 24.

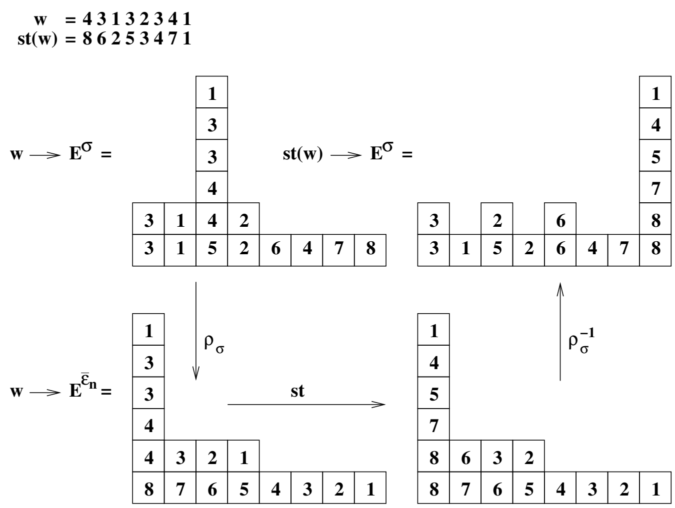


Figure 24: An example of the commutativity of standardization with the insertion algorithm with basement $\sigma = 3\ 1\ 5\ 2\ 6\ 4\ 7\ 8$.

By the same reasoning, we can show that the RSK algorithm for PBFs with basement σ also commutes with standardization. That is, suppose we are given an $\mathbb{N}$-valued $n \times n$ matrix A such that the sum of the entries of A is less than or equal to n. Then if $w_A = \begin{matrix} u_A \\ v_A \end{matrix}$ and

$$\Psi_\sigma(A) = \Psi_\sigma \begin{pmatrix} u_A \\ v_A \end{pmatrix} = (P^\sigma, Q^\sigma),$$

it will be the case that

$$\Psi_\sigma \begin{pmatrix} st(u_A) \\ st(v_A) \end{pmatrix} = (\rho_\sigma^{-1}(st(\rho(P^\sigma))), \rho_\sigma^{-1}(st(\rho(Q^\sigma)))).$$

7 Evacuation

The evacuation procedure on reverse semi-standard Young tableaux associates to each reverse SSYT T a new reverse SSYT $evac(T)$ through a deletion process coupled with jeu de taquin. Specifically, let T be a reverse SSYT with n cells whose largest entry is m and let a be the element in cell $(1, 1)$. Remove the entry a from T and apply jeu de taquin to create a new reverse SSYT, T', with $n-1$ cells. The skew shape $sh(T)/sh(T')$ therefore consists of one cell which is then filled with the complement, $m+1-a$, of a relative to m. Repeat this procedure with T' (but without changing the value of m) and continue until all of the cells from T have been evacuated and their complements relative to m have been placed into the appropriate locations in the diagram consisting of the union of all the one-celled skew shapes. This resulting diagram is a reverse semi-standard Young tableau called $evac(T)$.

We define an evacuation procedure on standard PBFs with basement σ as follows. Given a standard PBF F^σ with basement σ, we define $evac(F^\sigma) = \rho_\sigma^{-1}(evac(\rho_\sigma(F^\sigma)))$. That is, we first use the ρ_σ map to send F^σ to a reverse tableau $\rho_\sigma(F^\sigma)$. Then we apply the usual evacuation procedure to produce a reverse tableau $evac(\rho_\sigma(F^\sigma))$ and next apply ρ_σ^{-1} to map $evac(\rho(F^\sigma))$ back to a standard PBF with basement σ. We claim that in the special case where $\sigma = \epsilon_n$ is the identity, then we can define the evacuation procedure directly on the semi-standard PBF which will allow us to compute evacuation without using jeu de taquin.

Procedure 22. *Let F^{ϵ_n} be an arbitrary* PBF *of size n whose largest entry is m, and let R_i be the collection of entries appearing in the i^{th} row of F^{ϵ_n}, reading from bottom to top. Let e_1 be the largest element in the first row of F^{ϵ_n}, C_1 be the column containing e_1, and let h_1 be the height of C_1 in F^{ϵ_n}. Assign $m+1-e_1$ to row R_{h_1} in $evac(F^{\epsilon_n})$.Remove e_1 and shift the remaining entries in column C_1 down by one position so that there are no gaps in the column. Next rearrange the entries in the rows in the resulting figure according to the same procedure that we used in defining the $\rho_{\epsilon_n}^{-1}$ map to produce a* PBF $F_1^{\epsilon_n}$. *Repeat the procedure on the new diagram $F_1^{\epsilon_n}$. That is, let e_2 be the largest element in the first row of $F_1^{\epsilon_n}$, let C_2 be the column that contains e_2, and let h_2 be the height of column C_2 in $F_1^{\epsilon_n}$. Assign $m+1-e_2$ to row R_{h_2} in $evac(F^{\epsilon_n})$. Remove e_2 and shift the remaining entries in column C_2 down by one position so that there are no gaps in the column. Next rearrange the entries in the rows of the resulting figure according to same procedure that we used in defining the $\rho_{\epsilon_n}^{-1}$ map to produce a* PBF $F_2^{\epsilon_{n-2}}$. *Continue in this manner until all of the entries have been removed. The* PBF *$evac(F^{\epsilon_n})$ is produced by letting row i contain the complements of each entry relative to m (as described above) associated with a column of height i and applying the map $\rho_{\epsilon_n}^{-1}$ to send the resulting entries in the given rows to their appropriate places.*

See Figure 23 for an example of this procedure.

Theorem 23. *If F^{ϵ_n} is a* PBF, *then one can construct $evac(F^{\epsilon_n}) = \rho_{\epsilon_n}^{-1}(evac(\rho(F^{\epsilon_n})))$ by procedure 22.*

Proof. Let F be a PBF with basement ϵ_n and let $G = \rho_{\epsilon_n}^{-1}(F)$. Let $e_1 = F(1,1)$ so that e_1 is the largest element in the first row of F and hence it will be the largest element in the first row of G. Now consider the jeu de taquin path of the empty space created by the removal of e_1 from F. That is, in jeu de taquin, we move the empty space to cell $(2,1)$ and put $F(2,1)$ in cell (1,1) if $F(2,1)$ is defined and either $F(2,1) \geq F(1,2)$ or $F(1,2)$ is not defined. Otherwise, we put $F(1,2)$ in cell $(1,1)$ and move the empty space to cell $(1,2)$. In general, if the empty space is in cell (i,j), then we move the empty space to cell $(i+1,j)$ and put $F(i+1,j)$ into cell (i,j) if $F(i+1,j)$ is defined and either $F(i+1,j) \geq F(i,j+1)$ or $F(i,j+1)$ is not defined. Otherwise, we put $F(i,j+1)$ in cell (i,j) and move the empty space to cell $(i,j+1)$.

Now suppose in the evacuation of $e_1 = F(1,1)$, the path of the empty space ends in row s and that c_i is the right-most column involved in the jeu de taquin path in row i for $i = 1,\ldots,s$. Thus the jeu de taquin path involves cells $(1,1),\ldots,(c_1,1)$ in row 1 of F, cells $(c_1,2),\ldots,(c_2,2)$ in row 2 of F, cells $(c_2,3),\ldots,(c_3,3)$ in row 3 of F, etc.. If F_1 is the PBF with basement $\bar{\epsilon}_n$ that results from evacuating e_1, it follows that in F_1, each of the elements $F(c_i, i+1)$ will end up in row i of F_1 and all the other elements will be in the same row in F_1 as they were in F. We claim that in $G = \rho_{\epsilon_n}^{-1}(F)$, the column containing e_1 consists of $e_1, F(c_1,2), F(c_2,3),\ldots,F(c_{s-1},s)$, reading from bottom to top. Once we prove the claim, it will follow that in our direct evacuation of e_1 in G to produce $G_1^{\epsilon_n}$, the elements in row i of F_1 and $G_1^{\epsilon_n}$ are the same. But then $\rho(G_1^{\epsilon_n}) = F_1$ so that $\rho_{\epsilon_n}^{-1}(F_1) = G_1^{\epsilon_n}$ since the row sets of F_1 completely determine $\rho_{\epsilon_n}^{-1}(F_1)$. The theorem then easily follows by induction.

To prove the claim, note that the elements in the first row of F must all be distinct so that in constructing $\rho_{\epsilon_n}^{-1}(F)$, each element i in row 1 of F will be placed on column i. Now the fact that $(2,1),\ldots,(c_1,1)$ are in the jeu de taquin path means that $F(2,1) \geq F(1,2), F(3,1) \geq F(2,2),\ldots,F(c_1,1) \geq F(c_1-1,2)$. The fact that $F(c_1,2)$ is in the jeu de taquin path means that $F(c_1,2) > F(c_1+1,1)$ or $F(c_1+1,1)$ is not defined. It then follows that in constructing $\rho_{\epsilon_n}^{-1}(F)$, the elements $F(1,2),\ldots F(c_1-1,2)$ can be placed on the columns occupied by $F(2,1),\ldots,F(c_1,1)$ but not on top of any of the columns occupied by $F(c_1+1,1), F(c_1+2,1),\ldots$. Thus the $F(1,2),\ldots F(c_1-1,2)$ will be placed somewhere in the columns occupied by $F(2,1),\ldots,F(c_1,1)$. Thus when we place $F(c_1,2)$ in the left-most available column, it must go on top of e_1 since it can not go on top of any of the columns occupied by $F(c_1+1,1), F(c_1+2,1),\ldots$. Finally any elements strictly right of $(c_1, 2)$ in row 2 must be placed on top of columns occupied by elements strictly to the left of the column containing e_1 in row 1 of F. Now consider the construction of the third row of $\rho_{\epsilon_n}^{-1}(F)$. The elements $F(1,3), \ldots, F(c_1-1,3)$ can go on top of the elements $F(1,2), \ldots, F(c_1-1,2)$ since $F(i,3) \leq F(i,2)$ for all i for which both $F(i,3)$ and $F(i,2)$ are defined. Next the fact that $(c_1+1,2), \ldots, (c_3,2)$ are in the jeu de taquin path means that $F(c_1+1,2) \geq F(c_1,3), \ldots, F(c_2-1,3) \geq F(c_2,2)$. Thus $F(c_1,3), F(c_1+1,3), \ldots, F(c_2,3)$ can go on top of the elements $F(c_1+1,2), \ldots, F(c_2,2)$ in row two of $\rho_{\epsilon_n}^{-1}(F)$. The fact that $(c_2,3)$ is in the jeu de taquin path of e_1 in F means that $F(c_2,3) > F(c_2+1,2)$ so that none of $F(c_1+1,3), \ldots, F(c_2,3)$ can go on top of elements $F(c_2+1,2), F(c_2+2,2), \ldots$ in row two of $\rho_{\epsilon_n}^{-1}(F)$. Hence $F(1,3), \ldots, F(c_2-1,3)$ will be able go on top of elements $F(1,2), \ldots, F(c_1-1,2), F(c_1+1,2), \ldots, F(c_2,2)$ in row 2 of $\rho_{\epsilon_n}^{-1}(F)$ but they can not go on top of the elements $F(c_2+1,2), F(c_2+2,2), \ldots$ in row 2 of $\rho_{\epsilon_n}^{-1}(F)$. Hence it must be the case that $F(1,3), \ldots, F(c_2-1,3)$ end up on top of elements $F(1,2), \ldots, F(c_1-1,2), F(c_1+1,2), \ldots, F(c_2,2)$ in row 2 of $\rho_{\epsilon_n}^{-1}(F)$. Since $F(c_2,3)$ can not go on top of the elements $F(c_2+1,2), F(c_2+2,2), \ldots$ in row 2 of $\rho_{\epsilon_n}^{-1}(F)$, the only place left to place $F(c_2,3)$ is on top of column that contains e_1. Continuing on in this way establishes the claim. $\square$

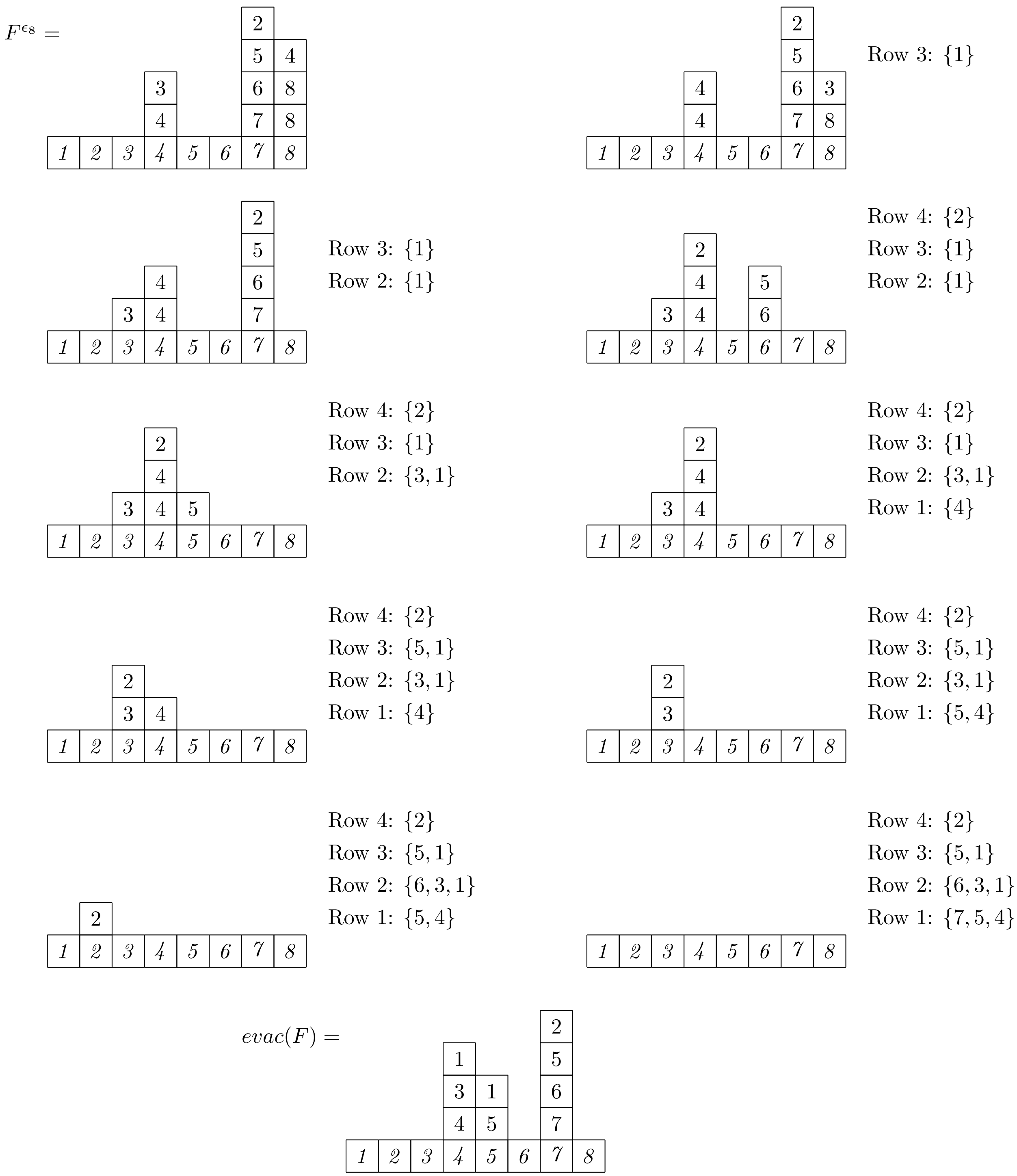


Figure 23: The evacuation procedure on a PBF with basement ϵ_8.

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