p-Rook Numbers and Cycle Counting in $C_p \wr S_n$

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Abstract

Cycle-counting rook numbers were introduced by Chung and Graham [7]. Cycle-counting q-rook numbers were introduced by Ehrenborg, Haglund, and Readdy [9] and cycle-counting q-hit numbers were introduced by Haglund [14]. Briggs and Remmel [4] introduced the theory of p-rook and p-hit numbers which is a rook theory model where the rook numbers correspond to partial permutations in $C_p \wr S_n$, the wreath product of the cyclic group C_p and the symmetric group S_n, and the hit numbers correspond to signed permutations in $C_p \wr S_n$. In this paper, we extend the cycle-counting q-rook numbers and cycle-counting q-hit numbers to the Briggs-Remmel model. In such a setting, we define a multivariable version of the cycle-counting q-rook numbers and cycle-counting q-hit numbers where we keep track of cycles of permutations and partial permutations of $C_p \wr S_n$ according to the signs of the cycles.

1 Introduction

We let $[n] = \{1, \ldots, n\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers, $\mathbb{P} = \{1, 2, \ldots\}$. A *board* is a subset of $\mathbb{P} \times \mathbb{P}$. We label the rows of $\mathbb{P} \times \mathbb{P}$ from

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bottom to top with $1, 2, 3, \ldots$, and the columns of $\mathbb{P} \times \mathbb{P}$ from left to right with $1, 2, 3, \ldots$, and let (i, j) denote the square in the i-th row and j-th column. Given $b_1, \ldots, b_n \in \mathbb{N}$, we let $F(b_1, \ldots, b_n)$ denote the board consisting of all the cells $\{(i, j) : 1 \leq i \leq n \ \& \ 1 \leq j \leq b_i\}$. If a board B is of the form $B = F(b_1, \ldots, b_n)$, then B is *skyline board* and if, in addition, $b_1 \leq b_2 \leq \cdots \leq b_n$, then B is a *Ferrers board*.

Given a board $B \subseteq [n] \times [n]$, we let

1. $\mathcal{N}_k(B)$ denote the set of all placements of k rooks in B such that no two rooks lie in the same row or column and

2. $\mathcal{C}_k(B)$ denote the set of placements of k rooks in B such that there is at most one rook in each column.

Elements of $\mathcal{N}_k(B)$ will be called rook placements and elements of $\mathcal{C}_k(B)$ will be called file placements. For $k = 1, \ldots, n$, we let $r_k(B) = |\mathcal{N}_k(B)|$ and $f_k(B) = |\mathcal{C}_k(B)|$. By convention, we set $r_0(B) = f_0(B) = 1$. We refer to $r_k(B)$ as the **k-th rook number of B** and $f_k(B)$ as the **k-th file number of B**.

Let S_n denote the symmetric group, i.e. the group of all permutations of $1, \ldots, n$ under composition. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we identify each $\sigma \in S_n$ with the rook placement $\{(\sigma_i, i) : i = 1, \ldots, n\}$ on $[n] \times [n]$. We let $\mathbb{F}_n$ denote the set of all functions $f : [n] \to [n]$. We will identify $f \in \mathbb{F}_n$ with the rook placement $\{(f(i), i) : i = 1, \ldots, n\}$ on $[n] \times [n]$. For example, if $\sigma = 2\ 3\ 1\ 5\ 4 \in S_n$ and f is the function given by $f(1) = 3$, $f(2) = 1$, $f(3) = 3$, $f(4) = 1$, and $f(5) = 4$, then the rook placement associated with σ is given on the left in Figure 1 and the file placement associated with f is given on the right in Figure 1. We let

$$\begin{aligned} H_{k,n}(B) &= |\{\sigma \in S_n : |\sigma \cap B| = k\}| \text{ and} \\ F_{k,n}(B) &= |\{f \in \mathbb{F}_n : |f \cap B| = k\}|. \end{aligned}$$

We shall refer to $H_{k,n}(B)$ as the **k-th hit number of B relative to** $[n] \times [n]$ and $F_{k,n}(B)$ as the **k-th fit number of B relative to** $[n] \times [n]$.

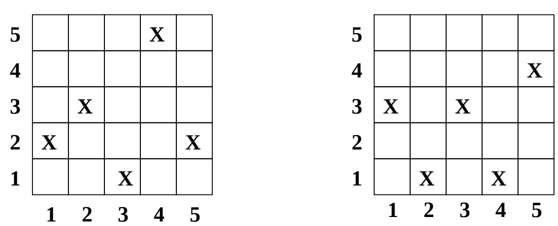


Figure 1: Rook placements associated with permutations and functions.

Kaplansky and Riordan [20] proved the following fundamental relationship between the rook numbers and the hit numbers of a board $B \subseteq [n] \times [n]$.

Theorem 1. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} H_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k. \tag{1}$$

Similarly, Miceli and Remmel [18] proved that

Theorem 2. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} F_{k,n}(B)x^k = \sum_{k=0}^{n} f_k(B)n^{n-k}(x-1)^k. \tag{2}$$

With each rook placement $P \in \mathcal{N}_k(B)$, we can associate a directed graph $G_P = ([n], E_P)$ where E_P is the set of (i,j) such that P has a rook in cell (i,j). Similarly, for each file placement $F \in \mathcal{C}_k(B)$, we can associate a directed graph $G_F = ([n], E_F)$ where E_F is the set of (i,j) such that F has a rook in cell (i,j). For example, Figure 2 for the graphs associated with rook and file placements in the $[6] \times [6]$ board. For any rook or file placement P, we let $\mathrm{cyc}(P)$ denote the number of cycles in the graph of P.

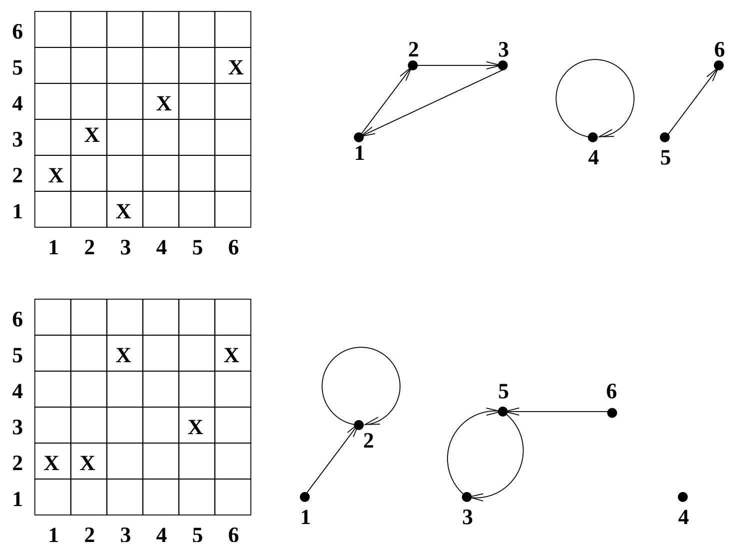


Figure 2: Graphs associated with rook and file placements.

For any board $B \subseteq [n] \times [n]$, we let

$$\begin{aligned} r_k(B,y) &= \sum_{P \in \mathcal{N}_k(B)} y^{\mathrm{cyc}(P)}, \\ f_k(B,y) &= \sum_{P \in \mathcal{C}_k(B)} y^{\mathrm{cyc}(P)}, \\ H_{k,n}(B,y) &= \sum_{\sigma \in S_n, |\sigma \cap B| = k} y^{\mathrm{cyc}(P)}, \text{ and} \\ F_{k,n}(B,y) &= \sum_{f \in \mathbb{F}_n, |f \cap B| = k} y^{\mathrm{cyc}(P)}. \end{aligned}$$

For $k \geq 1$, we let $(y)\uparrow_k = y(y+1)\cdots(y+k-1)$ and $(y)\downarrow_k = y(y-1)\cdots(y-k+1)$. We let $(y)\uparrow_0 = (y)\downarrow_0 = 1$. We then have the following analogues of Theorems 1 and 2.

Theorem 3. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} H_{k,n}(B,y)x^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k}(x-1)^k. \tag{3}$$

Theorem 4. *For any board $B \subseteq [n] \times [n]$,*

$$\sum_{k=0}^{n} F_{k,n}(B,y)x^k = \sum_{k=0}^{n} f_k(B,y)(y+n-1)^{n-k}(x-1)^k. \tag{4}$$

Proof. The proofs of both theorems are essentially the same. That is, replace x by $x+1$ in equations (3) and (4). Thus we must prove

$$\sum_{k=0}^{n} H_{k,n}(B,y)(x+1)^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} x^k \tag{5}$$

and

$$\sum_{k=0}^{n} F_{k,n}(B,y)(x+1)^k = \sum_{k=0}^{n} f_k(B,y)(y+n-1)^{n-k}x^k. \tag{6}$$

For (5), we consider configurations C which consist of a rook placement corresponding to a permutation $\sigma \in S_n$ where we circle some of the rooks that fall in $B \cap \sigma$. We then let $\mathrm{cyc}(C)$ denote the number of cycles in the graph of the underlying permutation of C and $\mathrm{circle}(C)$ denote the number of circled rooks in C. It is then easy to see that the left-hand side of (5) can be interpreted as counting $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all such configurations. The right-hand side of (5) can be interpreted as follows. First pick the circled rooks which corresponds to a placement $Q \in \mathcal{N}_k(B)$ for some k. Then we need to compute

$$A(Q,y) = \sum_{C} y^{\mathrm{cyc}(C)} \tag{7}$$

where the sum runs over all configurations whose set of circled rooks equals Q. But this sum is easy to compute. That is, let i be the first column that does not contain a rook in Q. Then there are $n-k$ rows in which to place a rook in column i that do not contain rooks in Q. We claim that there is exactly one row r where placing a rook in cell (i,r) completes a cycle in the graph of Q. That is, if there is no rook in Q which is in row i, then i is an isolated vertex in the graph of Q so adding a rook in cell (i,i) will give a loop on vertex i and hence increase the number of cycles by 1. Clearly in such a situation, placing a rook in cell (i,j) for $j \neq i$ cannot complete a cycle. If there is a rook in Q in row i, then there must be a maximal length path p in the graph of Q which ends in vertex i since there are no edges out of i in the graph of Q. If this path starts in vertex j, then there is no rook in row j in Q. Hence if we add a rook to cell (i,j), then we will complete a cycle. Clearly, adding a rook to any other row in column i will not complete a cycle in this case. Thus the placement of a rook in column i will contribute a factor of $(n-k-1+y)$ to $A(Q,y)$. But then we can repeat the argument for every placement Q' which arises from Q by adding a rook in the next empty column, say column i_1. That is, for each such Q', the addition of a rook in column i_1 will contribute a factor of $(y+n-k-2)$ to $A(Q,y)$. Continuing on in this way, we see that

$$A(Q,y) = (y+n-k-1)(y+n-k-2)\cdots(y) = (y)\uparrow_{n-k}.$$

Thus another way to sum $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all configurations is

$$\begin{aligned}
&\sum_{k=0}^{n} x^k \sum_{Q\in\mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)} A(Q,y)\\
&= \sum_{k=0}^{n} x^k \sum_{Q\in\mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)} (y)\uparrow_{n-k}\\
&= \sum_{k=0}^{n} x^k (y)\uparrow_{n-k} \sum_{Q\in\mathcal{N}_k} y^{\mathrm{cyc}(Q)}\\
&= \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} x^k.
\end{aligned}$$

The argument to prove (6) is similar. That is, we consider configurations C which consist of a file placement corresponding to a function $f:[n]\to[n]$. where we circle some of the rooks that fall in $B\cap f$. We then let $\mathrm{cyc}(C)$ denote the number of cycles in the graph of the underlying function of C and $\mathrm{circle}(C)$ denote the number of circled rooks in C. It is then easy to see that the left-hand side of (6) can be interpreted as counting $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all such configurations. The right-hand side of (6) can be interpreted as follows. First pick the circled rooks which corresponds to a placement $F\in\mathcal{C}_k(B)$ for some k. Then we need to compute

$$B(F,y) = \sum_{C} y^{\mathrm{cyc}(C)} \tag{8}$$

where the sum runs over all configurations whose set of circled rooks equals F. In this case, we see that each non-empty column contributes a factor of $n-1+y$ since there is no restriction on which rows we can place a rook in an given column. Thus $B(F,y)=(n-1+y)^{n-k}$. Thus another way to sum $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all configurations is

$$\begin{aligned}
&\sum_{k=0}^{n} x^k \sum_{F\in\mathcal{C}_k(B)} y^{\mathrm{cyc}(F)} B(F,y) \\
&= \sum_{k=0}^{n} x^k \sum_{F\in\mathcal{C}_k(B)} y^{\mathrm{cyc}(F)} (y+n-1)^{n-k} \\
&= \sum_{k=0}^{n} x^k (y+n-1)^{n-k} \sum_{F\in\mathcal{C}_k} y^{\mathrm{cyc}(F)} \\
&= \sum_{k=0}^{n} f_k(B,y)(y+n-1)^{n-k} x^k.
\end{aligned}$$

□

Chung and Graham [7] proved that for Ferrers boards $F(b_1,\ldots,b_n)\subseteq[n]\times[n]$, we have the following factorization theorem.

Theorem 5. *Let $B=F(b_1,\ldots,b_n)\subseteq[n]\times[n]$ be a Ferrers board. Then*

$$\prod_{i:b_i<i}(x+b_i-i+1)\prod_{i:b_i\geq i}(x+b_i-i+y)=\sum_{k=0}^{n} r_{n-k}(B,y)(x)\downarrow_k . \qquad (9)$$

We let

$$\begin{aligned}
[n]_q &= \frac{q^n-1}{q-1} = 1+\cdots q^{n-1}, \\
[n]_q! &= [1]_q[2]_q\cdots[n]_q, \text{ and} \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}
\end{aligned}$$

be the usual q-analogues of n, $n!$, and $\binom{n}{k}$. In general, we let $[x]_q=\frac{q^x-1}{q-1}$. Then for $k\geq 1$, we let $[x]_q\uparrow_k=[x]_q[x+1]_q\cdots[x+k-1]_q$ and $[x]_q\downarrow_k=[x]_q[x-1]_q\cdots[x-(k-1)]_q$. We let $[x]_q\uparrow_0=[x]_q\downarrow_0=1$.

In an unpublished paper, Ehrenborg, Haglund, and Readdy [9] defined a q-analogue of the cycle counting rook numbers $r_k(B,y,q)$ for Ferrers boards which generalized the q-analogue of the rook numbers for Ferrers boards introduced by Garsia and Remmel [11]. They proved the following generalization of Chung and Graham's theorem.

Theorem 6. *Let $B=F(b_1,\ldots,b_n)\subseteq[n]\times[n]$ be a Ferrers board. Then*

$$\prod_{i:b_i<i}[x+b_i-i+1]_q\prod_{i:b_i\geq i}[x+b_i-i+y]_q=\sum_{k=0}^{n} r_{n-k}(B,y,q)[x]_q\downarrow_k . \qquad (10)$$

Haglund [14] also extended the definition of the q-hit numbers of Garsia and Remmel [11] for Ferrers boards by defining q, x, y-hit numbers algebraically by the equation

$$\sum_{k=0}^{n} H_{k,n}(B, x, y, q) z^k \tag{11}$$
$$= \sum_{k=0}^{n} r_{n-k}(B, y, q)[x]_q \uparrow_k z^k \prod_{i=k+1}^{n} (1 - zq^{x+i-1}).$$

Haglund [14] developed several connections between formulas for the q, x, y-hit numbers and hypergeometric series. Later Butler [5] gave a combinatorial interpretation of $H_{k,n}(B, x, y, q)$ for Ferrers boards.

The main goal of this paper is to define the analogues of cycle-counting rook numbers, cycle-counting file numbers, and cycle-counting hit and fit numbers and their q-analogues relative to the group $C_p \wr S_n$ which is the wreath product of the cyclic group C_p of order p with the symmetric group S_n. In particular, we extend the combinatorics of cycle-counting rook numbers and cycle-counting hit numbers to the rook theory model of Briggs and Remmel [4] where rook placements correspond to partial signed permutations in $C_p \wr S_n$ and hit numbers correspond to signed permutations in $C_p \wr S_n$.

Let $\omega = e^{\frac{2\pi i}{p}}$. One can think of the group of $C_p \wr S_n$ as the group of matrices under matrix multiplication where the underlying set is the set of matrices that one can form by starting with an $n \times n$ permutation matrix M and replacing the 1's by powers of ω. Thus we can think of $C_p \wr S_n$ as the group of $p^n n!$ signed permutations where there are p signs, $1 = \omega^0$, ω, $\omega^2, \ldots, \omega^{p-1}$. We will usually write the signed permutations in either one-line notation or in disjoint cycle form. For example, if $\sigma \in C_3 \wr S_8$ is the map with $1 \to \omega 5$, $2 \to 8$, $3 \to \omega^2 3$, $4 \to \omega^2 1$, $5 \to 4$, $6 \to \omega^2 7$, $7 \to \omega 2$, and $8 \to \omega 6$, then in one-line notation,

$$\sigma = \omega 5 \;\; 8 \;\; \omega^2 3 \;\; \omega^2 1 \;\; 4 \;\; \omega^2 7 \;\; \omega 2 \;\; \omega 6,$$

whereas in disjoint cycle form,

$$\sigma = (\omega^2 1 \;\; \omega 5 \;\; 4)(\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)(\omega^2 3).$$

That is, in disjoint cycle form, to determine where i is being mapped, we ignore the sign on i and only consider the sign on the element to which it is mapped. Given an r-cycle $C = (\omega^{a_0} c_0, \ldots, \omega^{a_{r-1}} c_{r-1})$ in a signed permutation in $C_p \wr S_n$, we define $sgn(C) = \prod_{i=0}^{r-1} \omega^{a_i}$. Thus in our example,

$$\begin{aligned} &sgn((\omega^2 1 \;\; \omega 5 \;\; 4)) = 1, \\ &sgn((\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)) = \omega, \text{ and} \\ &sgn((\omega^2 3)) = \omega^2. \end{aligned}$$

Given $\sigma \in C_p \wr S_n$ we will write $\sigma(i)$ as $\varepsilon_i \sigma_i$ where $\sigma_i \in [n] = \{1, \ldots, n\}$ and where $\varepsilon_i = sgn(\sigma_i) \in \{1, \omega, \omega^2, \ldots, \omega^{p-1}\}$ is called the *sign* of σ_i. For each $1 \leq i \leq n$, we define $|\varepsilon_i \sigma_i| = \sigma_i$ and call this the *absolute value* of $\sigma(i)$.

Next we shall describe the rook model due to Briggs and Remmel [4] where the rook placements correspond to partial signed permutations in $C_p \wr S_n$ and the hit numbers count signed permutations in $C_p \wr S_n$. The idea of Briggs and Remmel was to start with the $[n] \times [n]$ board and subdivide each row into p subrows. We will call the resulting board B_n^p. For example, if $n = 6$ and $p = 3$, then B_6^3 is pictured in Figure 3. We shall refer to the rows of the original $[n]\times[n]$ board as levels and and label the levels with $1, \ldots, n$ from bottom to top. We label the columns with $1, \ldots, n$ from left to right. Finally, within each level, we label the sublevels from bottom to top with $1, \omega, \omega^2, \ldots, \omega^{p-1}$. We let (i, j, k) denote the square that is in i-th column, the j-th level, and in the sublevel labeled with ω^k.

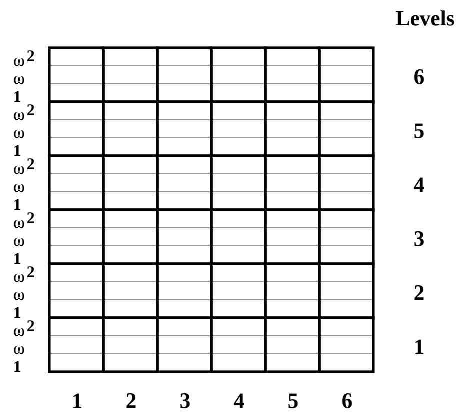


Figure 3: The board B_6^3.

In the Briggs-Remmel model, a *board* is a subset of B_n^p. Given $b_1, \ldots, b_n \in [pn]$, we let $F(b_1, \ldots, b_n)$ denote the board consisting of all the cells $\{(i, j, k) : 1 \leq i \leq n \ \& \ 1 \leq pj + k \leq b_i\}$. If a board B is of the form $B = F(b_1, \ldots, b_n)$, then we say that B is *skyline board* and if, in addition, $b_1 \leq b_2 \leq \cdots \leq b_n$ and $b_{i+1} \geq rp$ whenever $(r-1)p + 1 \leq b_i \leq rp$, then we say that B is a *Ferrers board*. Here the last condition for Ferrers boards in B_n^p says that whenever there are cells in level r in colum i, column $i+1$ must contain all the cells in the level r. Finally, we shall say that a board B is a *full board* if whenever, B contains a cell (i, j, k), then it must contain the cells (i, j, r) for $r = 0, \ldots, p-1$. Thus, example, a Ferrers board $F(b_1, \ldots, b_n)$ is a full board if and only b_i is a multiple of p for all $i = 1, \ldots, n$.

Given a board $B \subseteq B_n^p$, we let

1. $\mathcal{N}_k^p(B)$ denote the set of all placements of k rooks in B such that no two rooks lie in the same level or column and

2. $\mathcal{C}_k^p(B)$ denote the set of placements of k rooks in B such that there is at most one rook in each column.

Elements of $\mathcal{N}_k^p(B)$ will be called p-rook placements and elements of $\mathcal{C}_k^p(B)$ will be called p-file placements. For $k = 1, \ldots, n$, we let $r_k^p(B) = |\mathcal{N}_k^p(B)|$ and $f_k^p(B) = |\mathcal{F}_k(B)|$. By convention, we set $r_0^p(B) = f_0^p(B) = 1$. We refer to $r_k^p(B)$ as the **k-th p-rook number of B** and $f_k^p(B)$ as the **k-th p-file number of B**.

Given a signed permutation, $\sigma = \omega^{a_1}\sigma_1 \ldots \omega^{a_n}\sigma_n \in C_p \wr S_n$, we identify σ with the p-rook placement $\{(\sigma_i, i, a_i) : i = 1, \ldots, n\}$ on B_n^p. We let $\mathbb{F}_n^p$ denote the set of all functions $f : [n] \to [n] \times \{1, \omega, \ldots, \omega^{p-1}\}$. We will identify $f \in \mathbb{F}_n^p$ with the rook placement $\{(i, j, a_i) : f(j) = (i, a_i) \ \& \ j = 1, \ldots, n\}$ on B_n^p. For example, if $p = 3$ and $\sigma = 2\ \omega^2 3\ \omega 1\ \omega^2 5\ 4\ \omega 6 \in S_6$ and f is the function given by $f(1) = (3, 1)$, $f(2) = (1, \omega)$, $f(3) = (3, \omega^2)$, $f(4) = (1, 1)$, $f(5) = (4, 1)$, and $f(6) = (3, \omega)$ then the 3-rook placement associated with σ is given on the left in Figure 4 and the 3-file placement associated with f is given on the right in Figure 4. We let

$$\begin{aligned} H_{k,n}^p(B) &= |\{\sigma \in C_p \wr S_n : |\sigma \cap B| = k\}| \text{ and} \\ F_{k,n}^p(B) &= |\{f \in \mathbb{F}_n^p : |f \cap B| = k\}|. \end{aligned}$$

We shall refer to $H_{k,n}(B)$ as the **k-th p-hit number of B relative to B_n^p** and $F_{k,n}^p(B)$ as the **k-th p-fit number of B relative to B_n^p**.

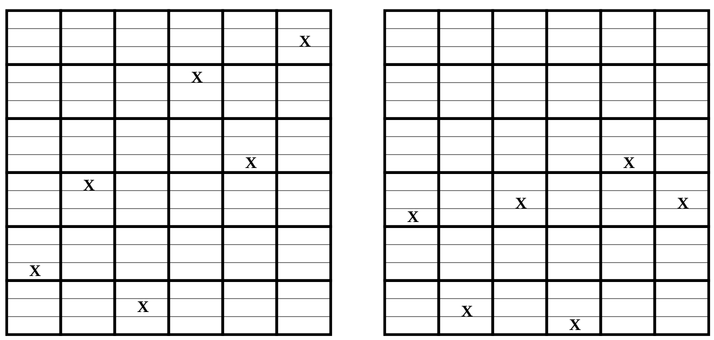


Figure 4: p-rook placements and p-file placements associated with signed permutations and signed functions.

With each p-rook placement $P \in \mathcal{N}_k^p(B)$, we can associate a directed graph $G_P = ([n], E_P)$ with labeled edges where E_P is the set of (i, j) such that P has a rook in cell (i, j, k) and we label the edge (i, j) with ω^k. Similarly, for each p-file placement $F \in \mathcal{C}_k^p(B)$, we can associate a directed graph $G_F = ([n], E_F)$ with labeled edges where E_F is the set of (i, j) such that F has a rook in cell (i, j, k) and we label the edge (i, j) with ω^k. For example, see Figures 5 and 6 for the graphs associated with 6-rook and 6-file placements in B_6^3. For any p-rook or p-file placement P, we let $\mathrm{cyc}_i(P)$ denote the number of cycles in the graph of P such that product of labels on the cycle is ω^i.

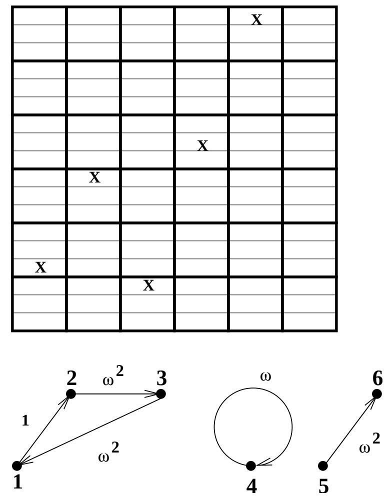


Figure 5: The graph associated with a 3-rook placement.

For any board $B \subseteq B_n^p$, we let

$$r_k^p(B, y_0, \ldots, y_{p-1}) = \sum_{P \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y^{\mathrm{cyc}_i(P)},$$

$$f_k^p(B, y_0, \ldots, y_{p-1}) = \sum_{P \in \mathcal{C}_k^p(B)} \prod_{i=0}^{p-1} y^{\mathrm{cyc}_i(P)},$$

$$H_{k,n}(B, y_0, \ldots, y_{p-1}) = \sum_{\sigma \in C_p \wr S_n, |\sigma \cap B| = k} \prod_{i=0}^{p-1} y^{\mathrm{cyc}_i(\sigma)}, \text{ and}$$

$$F_{k,n}(B, y_0, \ldots, y_{p-1}) = \sum_{f \in \mathbb{F}_n^p, |f \cap B| = k} \prod_{i=0}^{p-1} y^{\mathrm{cyc}_i(f)}.$$

The outline of the paper is as follows. In Section 2, we shall prove the analogues Theorems 3, 4, and 5 as well as give some examples of cycle-counting p-rook and cycle-counting p-file numbers that correspond to classical numbers like the Stirling numbers of the first and second kind and the Lah numbers. In Section 3, we shall develop q-analogues of the cycle-counting p-rook and cycle-counting p-file numbers and prove analogues of some of the results of Ehrenborg, Haglund, and Readdy [9], Haglund [14], and Butler [5] on the q-cycle-counting rook numbers and q, x, y-hit numbers.

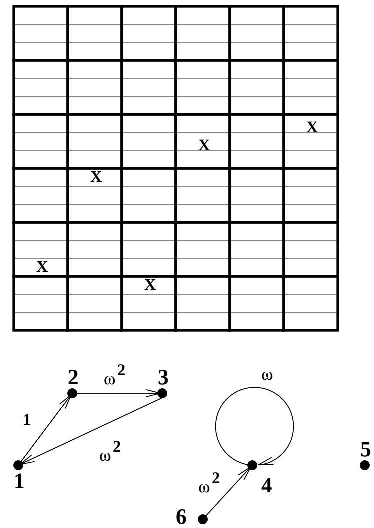


Figure 6: The graph associated with a 3-file placement.

2 Cycle-counting p-rook numbers, p-file, p-hit, and p-fit numbers.

We start this section by proving the analogues Theorems 3, 4, and 5 for the cycle-counting p-rook, p-file, p-hit, and p-fit numbers.

Suppose that $p \geq 2$. Then for $k \geq 1$, we let $(y) \uparrow_{k,p} = y(y+p)\cdots(y+p(k-1))$ and $(y) \downarrow_{k,p} = y(y-p)\cdots(y-p(k-1))$. We let $(y) \uparrow_{0,p} = (y) \downarrow_{0,p} = 1$. We then have the following analogues of Theorems 3 and 4.

Theorem 7. *For any $p \geq 2$ and any board $B \subseteq B_n^p$,*

$$\sum_{k=0}^{n} H_{k,n}^{p}(B, y_0, \ldots, y_{p-1})x^k \tag{12}$$
$$= \sum_{k=0}^{n} r_k^p(B, y_0, \ldots, y_{p-1})(y_0 + \cdots + y_{p-1}) \uparrow_{n-k,p} (x-1)^k.$$

Theorem 8. *For $p \geq 2$ and any board $B \subseteq B_N^p$,*

$$\sum_{k=0}^{n} F_{k,n}(B, y_0, \ldots, y_{p-1})x^k \tag{13}$$
$$= \sum_{k=0}^{n} f_k(B, y_0, \ldots, y_{p-1})(y_0 + \cdots + y_{p-1} + p(n-1))^{n-k}(x-1)^k.$$

Proof. Fix $p \geq 2$. Again the proofs of both theorems are essentially the same.

That is, replace x by $x+1$ in equations (12) and (13). Thus we must prove

$$\sum_{k=0}^{n} H_{k,n}^{p}(B, y_0, \ldots, y_{p-1})(x+1)^k \tag{14}$$
$$= \sum_{k=0}^{n} r_k^p(B, y_0, \ldots, y_{p-1})(y_0 + \cdots + y_{p-1}) \uparrow_{n-k,p} x^k$$

and

$$\sum_{k=0}^{n} F_{k,n}(B, y_0, \ldots, y_{p-1})(x+1)^k \tag{15}$$
$$= \sum_{k=0}^{n} f_k^p(B, y_0, \ldots, y_{p-1})(y_0 + \cdots + y_{p-1} + p(n-1))^{n-k} x^k.$$

For (14), we consider configurations C which consist of a rook placement corresponding to a signed permutation in $\sigma \in C_k \wr S_n$ where we circle some of the rooks that fall in $B \cap \sigma$. We then let $\mathrm{cyc}_i(C)$ denote the number of cycles of sign ω^i in the graph of the underlying signed permutation of C and $\mathrm{circle}(C)$ denote the number of circled rooks in C. It is then easy to see that the left-hand side of (14) can be interpreted as counting $x^{\mathrm{circle}(C)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all such configurations. The right-hand side of (14) can be interpreted as follows. First pick the circle rooks which corresponds to a placement $Q \in \mathcal{N}_k^p(B)$ for some k. Then we need to compute

$$A(Q, y_0, \ldots, y_{p-1}) = \sum_{C} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)} \tag{16}$$

where the sum runs over all configurations whose set of circled rooks equals Q.

Again this sum is easy to compute. That is, let i be the first column that does not contain a rook in Q. Then there are $n-k$ levels in which to place a rook in column i that do not contain rooks in Q. We claim that there is exactly one level r where placing a rook in cell (i,r,k) for any $0 \leq k \leq p-1$ completes a cycle in the graph of Q. That is, if there is no rook in Q which is in level i, then i is an isolated vertex in the graph of Q so adding a rook in cell (i,i,k) will give a loop on vertex i with label ω^k and hence increase the number of cycles with sign ω^k by 1. Clearly in such a situation, placing a rook in cell (i,j,k) for $j \neq i$ and $0 \leq k \leq p-1$ cannot complete a cycle. If there is a rook in Q in row i, then there must be a maximal length path p in the graph of Q which ends in vertex i since there are no edges out of i in the graph of Q. If this path starts in vertex j, then there is no rook in level j in Q. Hence if we add a rook to cell (i,j,k) for any $0 \leq k \leq p-1$, then we will complete a cycle. No matter what the labels are on the edges of the path from j to i in the graph corresponding to Q, there will be exactly one choice of k which results in the completed cycle having sign ω^i for any given $i \in \{0, \ldots, p-1\}$. Clearly, adding a rook to any other level in column i will not complete a cycle in this case. Thus the placement of a rook in column i will contribute a factor of $(y_0 + \cdots + y_{p-1} + p(n-k-1))$ to $A(Q, y_0, \ldots, y_{p-1})$. But then we can repeat the argument for every placement Q' which arises from Q by adding a rook in the next empty column, say column i_1. That is, for each such Q', the addition of a rook in column i_1 will contribute a factor of $(y_0 + \cdots + y_{p-1} + p(n-k-2))$ to $A(Q, y_0, \ldots, y_{p-1})$.

Continuing on in this way, we see that $A(Q,, y_0, \ldots, y_{p-1})$ equals

$$\begin{aligned}&(y_0 + \cdots + y_{p-1} + p(n-k-1))(y_0 + \cdots + y_{p-1} + p(n+k-2)) \cdots (y_0 + \cdots + y_{p-1}) \\ &= (y_0 + \cdots = y_{p-1} +) \uparrow_{n-k,p} .\end{aligned}$$

Thus another way to sum $x^{\mathrm{circle}(C)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all configurations is

$$\begin{aligned}&\sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} A(Q, y_0, \ldots, y_{p-1}) \\ &= \sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} (y + 0 + \cdots + y_{p-1}) \uparrow_{n-k,p} \\ &= \sum_{k=0}^{n} x^k (y + 0 + \cdots + y_{p-1}) \uparrow_{n-k,p} \sum_{Q \in \mathcal{N}_k} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} \\ &= \sum_{k=0}^{n} r_k^p(B, y_0, \ldots y_{p-1})(y_0 + \cdots + y_{p-1}) \uparrow_{n-k,p} x^k.\end{aligned}$$

The argument to prove (15) is similar. That is, we consider configurations C which consists of a file placement corresponding to a function $f : [n] \to [n] \times \{1, \omega, \ldots, \omega^{p-1}\}$, where we circle some of the rooks that fall in $B \cap f$. We then let $\mathrm{cyc}_i(C)$ denote the number of cycles of sign ω^i in the graph of the underlying function of C and $\mathrm{circle}(C)$ denote the number of circled rooks in C. It is then easy to see that the left-hand side of (15) can be interpreted as counting $x^{\mathrm{circle}(C)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all such configurations. The right-hand side of (15) can be interpreted as follows. First pick the circled rooks which corresponds to a placement $F \in \mathcal{C}_k^p(B)$ for some k. Then we need to compute

$$B(F, y_0, \ldots, y_{p-1}) = \sum_{C} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)} \tag{17}$$

where the sum runs over all configurations whose circled rooks equals F. In this case, we see that each non-empty column contributes a factor of $y_0 + \cdots + y_{p-1} + p(n-1)$ since there is no restriction on which levels we can place a rook in any given column. Thus $B(F, y_0, \ldots, y_{p-1}) = (y_0 + \cdots + y_{p-1} + p(n-1))^{n-k}$.

Thus another way to sum $x^{\mathrm{circle}(C)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all configurations is

$$\begin{aligned}
&\sum_{k=0}^{n} x^k \sum_{F \in \mathcal{C}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(F)} B(F, y_0, \ldots, y_{p-1}) \\
&= \sum_{k=0}^{n} x^k \sum_{F \in \mathcal{C}_k^p} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(F)} (y_0 + \cdots y_{p-1} + p(n-1))^{n-k} \\
&= \sum_{k=0}^{n} x^k (y_0 + \cdots y_{p-1} + p(n-1))^{n-k} \sum_{F \in \mathcal{C}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(F)} \\
&= \sum_{k=0}^{n} f_k^p(B, y_0, \ldots, y_{p-1})(y_0 + \cdots y_{p-1} + p(n-1))^{n-k} x^k.
\end{aligned}$$

□

Next we shall prove so-called factorization theorems of p-rook numbers and p-files numbers for full Ferrers boards $B \subseteq B_n^p$.

Theorem 9. *Let $p \geq 2$ and $B = F(b_1, \ldots, b_n)$ be a full Ferrers board contained in B_n^p. Then*

$$\begin{aligned}
&\prod_{i : b_i < pi} (x + b_i - p(i-1)) \prod_{i : b_i \geq pi} (x + b_i - pi + y_0 + \cdots + y_{p-1}) \qquad (18) \\
&= \sum_{k=0}^{n} r_{n-k}^p(B, y_0, \ldots, y_{p-1})(x) \downarrow_{k,p}
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{i : b_i < pi} (x + b_i) \prod_{i : b_i \geq pi} (x + b_i - p + y_0 + \cdots + y_{p-1}) \qquad (19) \\
&= \sum_{k=0}^{n} f_{n-k}^p(B, y_0, \ldots, y_{p-1}) x^k.
\end{aligned}$$

Proof. The assumption that B is a full board implies that b_i is divisible by p for all i. Since both sides of the (18) and (19) are polynomials in x of degree n, it is enough to prove that (18) and (19) holds for infinitely many integers.

First we shall show that (18) holds for infinitely many integers px where $x \in \mathbb{P}$. Given $x \in \mathbb{P}$, we let B_x denote the board, which results by adding x-levels of length n below B. For example, if $p = 3$, $B = (3, 6, 6, 6, 9, 9)$, and $x = 6$, then the board B_x is pictured in Figure 7. We call the boundary between B and the x-levels that we added below B the *bar*.

We let $\mathcal{N}_k^p(B_x)$ denote the set of all placements of k rooks in B_x such that there is at most one rook in each level and each column. Given a placement

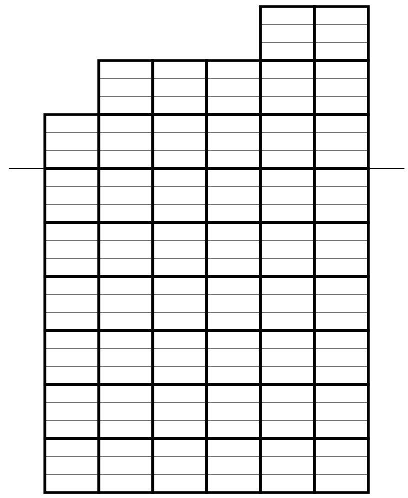


Figure 7: The board B_x.

$P \in \mathcal{N}_n^p(B_x)$, we let $wt(P) = \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P\cap B)}$. Then we claim that (18) where x is replaced by px arises from two different ways of computing

$$S(B, y_0, \ldots, y_{p-1}) = \sum_{P \in \mathcal{N}_n^p(B_x)} wt(P).$$

Next we prove a key lemma.

Lemma 10. *Suppose that $Q \in \mathcal{N}_t^p(B_x)$ is a p-rook placement of t rooks in the first $i-1$ columns of B_x. Let $D_i(Q)$ denote the set of all p-rook placements P that result from Q by adding a rook in column i. Then*

$$\sum_{P \in D_i(Q)} \prod_{i=0}^{p-1} y_i^{cyc_i(P)} =$$

$$\begin{cases} (b_i + px - p(t+1) + y_0 + \cdots + y_{p-1}) \prod_{i=0}^{p-1} y_i^{cyc_i(Q\cap B)} & \text{if } b_i \geq pi, \\ (b_i + px - pt + p) \prod_{i=0}^{p-1} y_i^{cyc_i(Q\cap B)} & \text{if } b_i < pi. \end{cases}$$

Proof. First we claim that there is exactly one level j above the bar such that placing a rook in a cell (i, j, k) will complete a cycle in the graph of $Q \cap B$ if $b_i \geq pi$ and there is no level j above the bar such that placing a rook in a cell (i, j, k) will complete a cycle in the graph of $Q \cap B$ if $b_i < pi$. That is, suppose that $b_i \geq pi$. Then if there is no rook in $Q \cap B$ which is in level i, then i is an isolated vertex in the graph of $Q \cap B$ so adding a rook in cell (i, i, k) will give a loop on vertex i with label ω^k and hence increase the number of cycles with sign ω^k by 1. Clearly in such a situation, placing a rook in cell (i, j, k) for $j \neq i$ and $0 \leq k \leq p-1$ cannot complete a cycle. If there is a rook in $Q \cap B$ in row i, then there must be a maximal length path p in the graph of $Q \cap B$ which ends in vertex i since there are no edges out of i in the graph of $Q \cap B$. If this path starts in vertex j, then $j \leq i \leq b_i/p$ and there is no rook in level j in $Q \cap B$ above the bar. Hence if we add a rook to cell (i,j,k) for any $0 \leq k \leq p-1$, then it will complete a cycle. No matter what the labels are on the edges of the path from j to i in the graph corresponding to Q, there will be exactly one choice of k which result in the completed cycle having sign ω^i for any given $i \in \{0,\ldots,p-1\}$. In such a situation, we will call the level j such that adding a rook in a cell (i,j,k) completes a cycle, the *special level relative to* Q. It easily follows that in this case,

$$\sum_{P \in D_i(Q)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = (b_i + px - p(t+1) + y_0 + \cdots y_{p-1}) \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q\cap B)}.$$

Alternatively, if $b_i < pi$, then we must have that $b_1 \leq \cdots \leq b_{i-1} \leq p(i-1)$ since we are assuming that B is full Ferrers board. This implies that there can be no edge which ends in i in the graph of $Q \cap B$. Hence i is an isolated vertex in the graph $Q \cap B$. Thus placing a rook in cell (i,j,k) where $j < i$ can not create a new cycle. Thus it easily follows that in this case,

$$\sum_{P \in D_i(Q)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = (b_i + px - pt) \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q\cap B)}.$$

□

Now think of adding rooks column by column starting from the left to form an element $P \in \mathcal{N}_n^p(B_x)$. In the first column, we have $b_i + px$ choices. If $b_1 \geq p$, then if we add a rook in cell $(1,1,k)$ then we create a cycle of sign ω^k and we do not create a cycle otherwise. Thus the first column will contribute a factor of $(px + b_1 - p + y_0 + \cdots + y_{p-1})$ if $b_1 \geq p$ and a factor $px + b_1$ otherwise. Next if we start with a placement $Q \in \mathcal{N}_{i-1}^p(B_x)$ of $i-1$ rooks in the first $i-1$ columns of B_x, then we will have $px + b_i - p(i-1)$ cells to add a rook in column i. By Lemma 10, our choices for placing a rook in these $px + b_i - p(i-1)$ cells will contribute a factor of $(px + b_i - pi + y_0 + \cdots + y_{p-1})$ if $b_i \geq pi$ and will contribute a factor of $(px + b_i - p(i-1))$ otherwise. Thus it follows that

$$S(y_0,\ldots,y_{p-1}) = \prod_{i:b_i<pi} (px + b_i - p(i-1)) \prod_{i:b_i \geq pi} (px + b_i - pi + y_0 + \cdots + y_{p-1}).$$

Next suppose that we fix a p-rook placement $Q \in \mathcal{N}_{n-k}^p(B)$ of $n-k$ rooks above the bar. Then we want to compute

$$B_Q = \sum_{P \in \mathcal{N}_n^p(B_x): P \cap B = Q} wt(P). \tag{20}$$

In this case, there will be k columns below the bar which do not contain rooks in Q. If the columns are $1 \leq i_1 < \cdots < i_k \leq n$, then it is easy to see that we have px choices to place a rook below the bar in column i_1. Once we have placed a rook in column i_1 below the bar, we will have $px - p$ choices to add a rook below the bar in column i_2. Continuing on in this way it is easy to see that we have $(px)(px-p)\cdots(px-p(k-1)) = (px)\downarrow_{k,p}$ ways to extend Q to a placement in $\mathcal{N}_n^p(B_x)$. By definition, the weight of any such placement P is $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}$. Thus

$$\begin{aligned} S(y_0,\ldots,y_{p-1}) &= \sum_{k=0}^{n} \sum_{Q\in\mathcal{N}_{n-k}^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} (px)\downarrow_{k,p} \\ &= \sum_{k=0}^{n} (px)\downarrow_{k,p} \sum_{Q\in\mathcal{N}_{n-k}^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} \\ &= \sum_{k=0}^{n} r_{n-k}^p(B,y_0,\ldots,y_{p-1})(px)\downarrow_{k,p}. \end{aligned}$$

The proof that (19) holds for px for infinitely many $x \in \mathbb{P}$ is essentially the same. That is, we let $\mathcal{C}_k^p(B_x)$ denote the set of all placements of k rooks in B_x such that there is at most one rook in each column. Given a placement $F \in \mathcal{C}_n^p(B_x)$, we let $wt(P) = \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(F\cap B)}$. Then we claim that (19) where x is replaced by px arises from two different ways of computing

$$T(B,y_0,\ldots,y_{p-1}) = \sum_{P\in\mathcal{C}_n^p(B_x)} wt(P).$$

By essentially the same argument that we used to prove Lemma 10, we can prove the following.

Lemma 11. *Suppose that $Q \in \mathcal{C}_t^p(B_x)$ is a p-file placement of t rooks in the first $i-1$ columns of B_x. Let $E_i(Q)$ denote the set of all p-file placements P that result from Q by adding a rook in column i. Then*

$$\sum_{P\in E_i(Q)} \prod_{i=0}^{p-1} y_i^{cyc_i(P)} = \begin{cases} (b_i + px - p + y_0 + \cdots y_{p-1}) \prod_{i=0}^{p-1} y_i^{cyc_i(Q\cap B)} & \text{if } b_i \geq pi, \\ (b_i + px) \prod_{i=0}^{p-1} y_i^{cyc_i(Q\cap B)} & \text{if } b_i < pi. \end{cases}$$

This given, we can argue as above that as we add rooks in columns from left to right that the i-th will give a contribution of $(px + b_i)$ if $b_i < pi$ and a contribution of $px + b_i - p + y_0 + \cdots + y_{p-1}$ if $b_i \geq pi$. Thus

$$T(B,y_0,\ldots,y_{p-1}) = \prod_{i:b_i<pi} (px+b_i) \prod_{i:b_i\geq pi} (px + b_i - p + y_0 + \cdots + y_{p-1}).$$

Next suppose that we fix a p-file placement $Q \in \mathcal{C}^p_{n-k}(B)$ of $n-k$ rooks above the bar. Then we want to compute

$$D_Q = \sum_{P \in \mathcal{C}^p_n(B_x): P \cap B = Q} wt(P). \tag{21}$$

In this case, there will be k columns below the bar which do not contain rooks in Q. In each such column, we have px choices to place a rook below the bar. Thus we have $(px)^k$ ways to extend Q to a placement in $\mathcal{C}^p_n(B_x)$. By definition, the weight of any such placement P is $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}$. Thus

$$\begin{aligned} T(y_0,\ldots,y_{p-1}) &= \sum_{k=0}^{n} \sum_{Q \in \mathcal{C}^p_{n-k}(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} (px)^k \\ &= \sum_{k=0}^{n} (px)^k \sum_{Q \in \mathcal{C}^p_{n-k}(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} \\ &= \sum_{k=0}^{n} f^p_{n-k}(B, y_0, \ldots, y_{p-1})(px)^k. \end{aligned}$$

□

Classical combinatorial numbers such as the Stirling numbers of the first and second kind and the Lah numbers have nice rook theory interpretations. In the next two subsections, we shall consider the analogues of such numbers for cycle-counting p-rook and p-file numbers.

2.1 Cycle counting p-Stirling numbers

Let $s_{n,k}$ and $S_{n,k}$ denote the Stirling numbers of the first and second kind, respectively. They are defined by the equations

$$\begin{aligned} (x)\downarrow_n &= \sum_{k=1}^{n} s_{n,k} x^k \text{ and} \\ x^n &= \sum_{k=1}^{n} S_{n,k} (x)\downarrow_k . \end{aligned}$$

The numbers $S_{n,k}$ and $c_{n,k} = (-1)^{n-k} s_{n,k}$ have nice rook theory interpretations. That is, let $St_n = F(0,1,\ldots,n-1)$ be the staircase board, then $S_{n,k} = r_{n-k}(St_n)$ and $c_{n,k} = f_{n-k}(St_n)$. For the p-rook and p-file numbers, the obvious analogue of the staircase board is $St^p_n = F(0,p,2p,\ldots,(n-1)p)$. In that case (18) and (19) become

$$(x)\uparrow_{n,p} \quad = \quad \sum_{k=1}^{n} f^p_{n-k}(St^p_n, y_0, \ldots, y_{p-1})x^k \text{ and} \tag{22}$$

$$x^n \quad = \quad \sum_{k=1}^{n} r^p_{n-k}(St^p_n, y_0, \ldots, y_{p-1})(x)\downarrow_{k,p}. \tag{23}$$

Because the height of the i-th column of St^p_n is less than pi for all i, no cycle counting is involved in the definitions of $f^p_{n-k}(St^p_n, y)$ and $r^p_{n-k}(St^p_n, y)$ so that they are independent of $(y_0, \ldots, y_{p-1})$. Thus if we let

$$\begin{aligned} S^p_{n,k} &= r^p_{n-k}(St^p_n, y_0, \ldots, y_{p-1}) \text{ and} \\ s^p_{n,k} &= (-1)^{n-k} f^p_{n-k}(St^p_n, y_0, \ldots, y_{p-1}), \end{aligned}$$

then these are the p-Stirling numbers of the first and second kind as defined by Briggs and Remmel [4]. They satisfy the following recursions:

$$S^p_{n+1,k} = S^p_{n,k-1} + pkS^p_{n,k} \tag{24}$$

with initial conditions $S^p_{0,0} = 1$ and $S^p_{n,k} = 0$ if $k < 0$ or $k > n$ and

$$s^p_{n+1,k} = s^p_{n,k-1} - pns^p_{n,k} \tag{25}$$

with initial conditions $s^p_{0,0} = 1$ and $s^p_{n,k} = 0$ if $k < 0$ or $k > n$. We note that Briggs and Remmel actually defined p,q-analogues of $S^p_{n,k}$ which are special cases of generalized Stirling numbers of the first and second defined by Remmel and Wachs [19].

A more interesting case from the point of view of the cycle counting is to consider the board $\tilde{St}_n = F(p, 2p, \ldots, np)$. Let

$$\tilde{S}^p_{n,k}(y_0, \ldots, y_{p-1}) = r^p_{n-k}(\tilde{St}^p_n, y_0, \ldots, y_{p-1})$$

and

$$\tilde{c}^p_{n,k}(y_0, \ldots, y_{p-1}) = f^p_{n-k}(\tilde{St}^p_n, y_0, \ldots, y_{p-1}).$$

In that case, (18) and (19) become

$$(x + y_0 + \cdots + y_{p-1})\uparrow_{n,p} = \sum_{k=1}^{n} \tilde{c}^p_{n,k}(y_0, \ldots, y_{p-1})x^k \tag{26}$$

and

$$(x + y_0 + \cdots + y_{p-1})^n = \sum_{k=1}^{n} \tilde{S}^p_{n,k}(y_0, \ldots, y_{p-1})(x)\downarrow_{k,p}. \tag{27}$$

Replacing x by $-x$ in (26) and then multiplying by $(-1)^n$ gives

$$(x - (y_0 + \cdots + y_{p-1}))\downarrow_{n,p} = \sum_{k=1}^{n} (-1)^{n-k} \tilde{c}^p_{n,k}(y_0, \ldots, y_{p-1})x^k \tag{28}$$

Thus if we let

$$\tilde{s}^p_{n,k}(y_0,\ldots,y_{p-1}) = (-1)^{n-k}\tilde{c}^p_{n,k}(y_0,\ldots,y_{p-1})$$

and replace x by $x+y_0+\cdots+y_{p-1}$ in (28), we obtain that

$$(x)\downarrow_{n,p}= \sum_{k=1}^{n} \tilde{s}^p_{n,k}(y_0,\ldots,y_{p-1})(x+y_0+\cdots+y_{p-1})^k. \tag{29}$$

Comparing (27) and (29), it is easy to see that the matrices $||\tilde{S}^p_{n,k}(y_0,\ldots,y_{p-1})||_{n,k\geq 0}$ and $||\tilde{s}^p_{n,k}(y_0,\ldots,y_{p-1})||_{n,k\geq 0}$ are inverses of each other.

We claim that we have the following recursions:

$$\begin{aligned} &\tilde{S}^p_{n+1,k}(y_0,\ldots,y_{p-1}) \\ &= S^p_{n,k-1}(y_0,\ldots,y_{p-1}) + (pk+y_0+\cdots y_{p-1})\tilde{S}^p_{n,k}(y_0,\ldots,y_{p-1}) \end{aligned} \tag{30}$$

with initial conditions $S^p_{0,0}(y_0,\ldots,y_{p-1}) = 1$ and $S^p_{n,k}(y_0,\ldots,y_{p-1}) = 0$ if $k<0$ or $k>n$ and

$$\begin{aligned} &c^p_{n+1,k}(y_0,\ldots,y_{p-1}) \\ &= c^p_{n,k-1}(y_0,\ldots,y_{p-1}) + (pn+y_0+\cdots y_{p-1})c^p_{n,k}(y_0,\ldots,y_{p-1}) \end{aligned} \tag{31}$$

with initial conditions $c^p_{0,0}(y_0,\ldots,y_{p-1}) = 1$ and $c^p_{n,k}(y_0,\ldots,y_{p-1}) = 0$ if $k<0$ or $k>n$. Both recursions have the same proof. That is, to prove (30), we partition the p-rook placements corresponding to $\mathcal{N}^p_{n+1-k}(\tilde{S}t_{n+1})$ by whether they have a rook in the last column or not. That is, it is easy to see that those rook placements that have no rook in the last column correspond to rook placements in $\mathcal{N}^p_{n+1-k}(\tilde{S}t_n)$ and hence

$$\begin{aligned} &\sum_{\substack{P \text{ has no rook in the last column} \\ P\in\mathcal{N}^p_{n+1-k}(\tilde{S}t_{n+1})}} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} \\ &= \sum_{P\in\mathcal{N}^p_{n+1-k}(\tilde{S}t_n)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = \tilde{S}^p_{n,k-1}(y_0,\ldots,y_{p-1}). \end{aligned}$$

If $Q\in\mathcal{N}^p_{n-k}(\tilde{S}t_n)$ of $n-k$ rooks in the first n columns, then there will be $n+1-(n-k)$ levels in which we can put a rook in the last column to extend Q to a placement P in the last column. If P comes from placing a rook in a cell $(n+1,n+1,i)$, then we create a loop labeled with ω^i so that

$$\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = y_i \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}.$$

If P comes from placing a rook in a cell $(i, n+1, k)$ with $i \leq n$, then it is easy to see that we cannot complete a cycle in the graph of Q so that

$$\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}.$$

It follows that for each such Q, the sum of $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)}$ over P's such that P extends Q by adding a rook in column n is

$$(pk + y_0 + \cdots y_{p-1}) \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}.$$

Thus it follows that

$$\sum_{\substack{P \text{ has a rook in the last column} \\ P \in \mathcal{N}^p_{n+1-k}(\tilde{S}t_{n+1})}} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)}$$
$$= (pk + y_0 + \cdots y_{p-1}) \sum_{Q \in \mathcal{N}^p_{n-k}(\tilde{S}t_n)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}$$
$$= (pk + y_0 + \cdots + y_{p-1}) \tilde{S}^p_{n,k}(y_0, \ldots, y_{p-1}).$$

The recursion (31) is proved in a similar way by considering what happens when we partition the p-file placements in $\mathcal{C}^p_{n+1-k}(\tilde{S}t_{n+1})$ according to whether they have a rook in the last column or not.

2.2 Cycle counting p-Lah numbers

The Lah numbers $L_{n,k}$ are defined by the equation,

$$(x) \uparrow_n = \sum_{k=1}^{n} L_{n,k} (x) \downarrow_k .$$

They can also be defined by the following recursion

$$L_{n+1,k} = L_{n,k-1} + (n+k) L_{n,k} \tag{32}$$

with initial conditions $L_{0,0} = 1$ and $L_{n,k} = 0$ if $k < 0$ or $k > n$. The $L_{n,k}$'s have a nice rook theory interpretation, namely, $L_{n,k} = r_{n-k}(\mathcal{L}_n)$ where $\mathcal{L}_n$ is the Ferrers board consisting of n columns of height $n-1$, see [11]. From this interpretation, it is easy to see that

$$L_{n,k} = \frac{(n-1)!}{(k-1)!} \binom{n}{k}. \tag{33}$$

That is, to create a rook placement of $n-k$ rooks in $\mathcal{L}_n$, we first pick the $n-k$ columns that will contain the rooks. We can do this in $\binom{n}{n-k} = \binom{n}{k}$ ways. Then we have to place the rooks in these columns starting from the left. We clearly have $n-1$ choices where to put a rook in the left most column, then $n-1-1$ ways to place a rook in the next column, etc. Thus we will have $(n-1)\downarrow_{n-k} = \frac{(n-1)!}{(k-1)!}$ ways to place the rooks in the $n-k$ columns that we chose.

For the obvious cycle-counting analogue of the $L_{n,k}$'s for $C_p \wr S_n$, consider the Ferrers board $\mathcal{L}_n^p$ which consists of n columns of height $p(n-1)$. We let

$$L^p_{n,k}(y_0,\dots y_{p-1}) = r^p_{n-k}(\mathcal{L}^p_n, y_0,\dots,y_{p-1}). \tag{34}$$

In this case, (18) becomes

$$x\cdot(x+y_0+\cdots+y_{p-1})\uparrow_{n-1,p} = \sum_{k=1}^{n} L^p_{n,k}(y_0,\dots y_{p-1})(x)\downarrow_{k,p}. \tag{35}$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{n+1} L^p_{n+1,k}(y_0,\dots y_{p-1})(x)\downarrow_{k,p}\\
&= x\cdot(x+y_0+\cdots+y_{p-1})\uparrow_{n,p}\\
&= (x+y_0+\cdots+y_{p-1}+p(n-1))x\cdot(x+y_0+\cdots+y_{p-1})\uparrow_{n-1,p}\\
&= (x+y_0+\cdots+y_{p-1}+p(n-1))\sum_{k=1}^{n} L^p_{n,k}(y_0,\dots y_{p-1})(x)\downarrow_{k,p}\\
&= \sum_{k=1}^{n} L^p_{n,k}(y_0,\dots y_{p-1})(x)\downarrow_{k,p}(x-kp+y_0+\cdots+y_{p-1}+p(n+k-1))\\
&= \sum_{k=1}^{n} L^p_{n,k}(y_0,\dots y_{p-1})(x)\downarrow_{k+1,p}\\
&+ \sum_{k=1}^{n} L^p_{n,k}(y_0,\dots y_{p-1})(x)\downarrow_{k,p}(y_0+\cdots+y_{p-1}+p(n+k-1)).
\end{aligned}$$

It thus follows that

$$\begin{aligned}
&L^p_{n+1,k}(y_0,\dots y_{p-1}) \\
&= L^p_{n,k-1}(y_0,\dots y_{p-1}) + (y_0+\cdots+y_{p-1}+p(n+k-1))L^p_{n,k}(y_0,\dots y_{p-1}).
\end{aligned} \tag{36}$$

We also have an analogue of the (33) in this case. That is, we want to compute

$$\sum_{P\in\mathcal{N}^p_{n-k}(\mathcal{L}^p_n)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)}.$$

We divide the p-rook placements in $\mathcal{N}^p_{n-k}(\mathcal{L}^p_n)$ into two sets, N_1 consisting of those p-rook placements with no rook in the last column and N_2 consisting of those p-rook placements that have a rook in the last column. For N_1, there are $\binom{n-1}{n-k} = \binom{n-1}{k-1}$ ways to choose the $n-k$ columns in which we are going to place the rooks. If $i \leq n-1$, then the height of the i-th column is greater than or equal to pi. Hence, we can use Lemma 10 to argue that as we place the rooks in the columns from left to right, the sum of $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)}$ over the possible placements in the $n-k$ columns that we choose is

$$(p(n-2)+y_0+\cdots+y_{p-1})(p(n-3)+y_0+\cdots+y_{p-1})\cdots(p(k-1)+y_0+\cdots+y_{p-1}).$$

Thus

$$\sum_{P \in N_1} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = \binom{n-1}{k-1} (p(k-1)+y_0+\cdots+y_{p-1}) \uparrow_{n-k,p} .$$

For N_2, there are $\binom{n-1}{n-k-1} = \binom{n-1}{k}$ ways to choose the columns in which we are going to place the rooks in the first $n-1$ columns. As above, the sum of $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)}$ over the possible placements in the $n-k$ columns that we choose is

$$(p(n-2)+y_0+\cdots+y_{p-1})(p(n-3)+y_0+\cdots+y_{p-1})\cdots(pk+y_0+\cdots+y_{p-1}).$$

Once we place these rooks, we still have to place a rook in the last column. However, the height of the last column in $\mathcal{L}_n^p$ is $(n-1)p < np$. Thus by Lemma 10, the factor contributed by placing the rook in the last column in the $n-1-(n-k-1) = k$ levels which are possible is pk. Thus

$$\sum_{P \in N_2} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} = \binom{n-1}{k} (pk)(pk+y_0+\cdots+y_{p-1}) \uparrow_{n-k-1,p} .$$

Hence,

$$\begin{aligned} L_{n,k}^p &= \binom{n-1}{k-1} (p(k-1)+y_0+\cdots+y_{p-1}) \uparrow_{n-k,p} \\ &\quad + \binom{n-1}{k} (pk)(pk+y_0+\cdots+y_{p-1}) \uparrow_{n-k-1,p} . \end{aligned}$$

3 Q-analogues of the cycle counting p-rook and p-hit numbers

In this section, we shall develop q-analogues of cycle counting p-rook numbers and cycle counting p-hit numbers.

First we shall recall the definitions of the q-analogues of the p-rook and p-hit numbers as defined by Briggs and Remmel [4]. Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board contained in B_n^p. A rook in cell (i, j, k) is said to *rook cancel* all cells in level j that lie strictly its right, all cells that lie directly below it, and itself. Then for any given $P \in \mathcal{N}_k^p(B)$, we let $inv_B(P)$ equal the number of uncancelled cells in B. For example, in Figure 8 we have pictured a placement in $B = F(6, 9, 12, 12, 15, 15) \subseteq B_6^3$ and we have put dots in cells which are rook cancelled by rooks in P. Thus $inv_B(P) = 30$ as there is a total of 30 squares which are not rook cancelled by rooks in P.

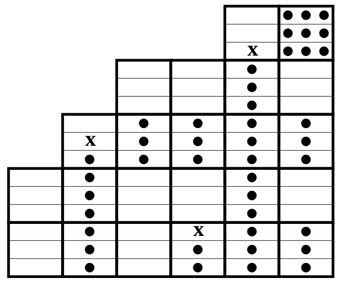


Figure 8: An example of rook cancellation.

Suppose that $p \geq 2$. Then for $k \geq 1$, we let

$$\begin{aligned} [y]_q \uparrow_{k,p} &= [y]_q[y+p]_q \cdots [y+p(k-1)]_q \text{ and} \\ [y]_q \downarrow_{k,p} &= [y]_q[y-p]_q \cdots [y-p(k-1)]_q. \end{aligned}$$

We let $[y]_q \uparrow_{0,p} = [y]_q \downarrow_{0,p} = 1$. Then for any Ferrers board $B = F(b_1, \ldots, b_n) \subseteq B_n^p$, Briggs and Remmel defined

$$r_k^p(B, q) = \sum_{P \in \mathcal{N}_k^p} q^{inv(P)} \tag{37}$$

and

$$\sum_{k=0}^{n} H_{k,n}(B, q)x^k = \sum_{k=0}^{n} r_k^p(B, q)[p(n-k)]_q \downarrow_{n-k,p} \prod_{\ell=n-k+1}^{n} (x - q^{p\ell}). \tag{38}$$

Briggs and Remmel [4] then proved the following two theorems.

Theorem 12. *Let $B = F(b_1, \ldots, b_n) \subseteq B_n^p$ be a Ferrers board. Then*

$$\prod_{i=1}^{n} [x + b_i - p(i-1)] = \sum_{k=0}^{n} r_{n-k}^p(B, q)[px] \downarrow_{k,p} . \tag{39}$$

Theorem 13. *Let $B = F(b_1, \ldots, b_n) \subseteq B_n^p$ be a Ferrers board. Then $H_{k,n}(B, q)$ is a polynomial in q with non-negative integer coefficients for all $k = 0, \ldots, n$.*

In fact, Briggs and Remmel proved p, q-analogues of Theorems 12 and 13 but we shall not concern ourselves with p, q-analogues in this paper.

Given a Ferrers board $B = F(b_1, b_2, \ldots, b_n) \subseteq B_n^p$, we will also use the notation $B = B(h_1^p, d_1; \ldots; h_t^p, d_t)$ which uses the step heights and depths as pictured in Figure 3.

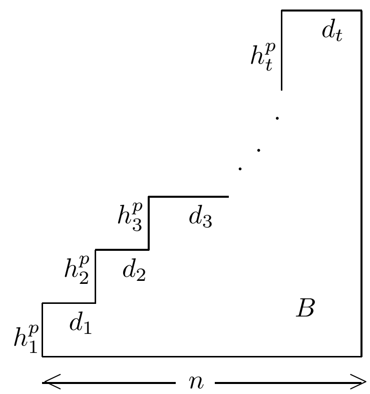


Figure 9: The Ferrers board $B = B(h_1^p, d_1; \ldots ; h_t^p, d_t)$

We define the q-analogue of the cycle-counting p-rook number by

$$r_k^p(B, q, y_0, \ldots, y_{p-1}) = \sum_{P \in \mathcal{N}_k^p(B)} \left(\prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P)} \right) q^{\mathrm{inv}(P) + \sum_{j=0}^{p-1} (y_j - 1) E_j(P)}, \tag{40}$$

where

inv(P) is the number of uncanceled cells (considering one sub level as one cell) when a rook cancels all the cells below it, and all the cells to the right in the same level with the rook,

$E_j(P)$ is the number of i's such that $b_i \geq i$ and there is no rook from P in column i on or above $s_i^j(P)$, where

$s_i^j(P)$, for i such that $b_i \geq pi$, is the unique sub level which, considering only rooks from P in column 1 through $i-1$ of B, completes the ω_j cycle.

Then we have the following q-analogue of the factorization theorem.

Theorem 14. *Let $B = F(b_1, \ldots, b_n)$ be a full Ferrers board contained in B_n^p.*

$$\prod_{i:b_i<pi} [px + b_i - p(i-1)]_q \prod_{i:b_i \geq pi} [px + b_i - pi + y_0 + \cdots + y_{p-1}]_q \tag{41}$$
$$= \sum_{k=0}^{n} r_{n-k}^p (B, q, y_0, \ldots, y_{p-1}) [px]_q \downarrow_{k,p} .$$

Proof. It is not difficult to show that it is enough to prove (41) holds whenever $x, y_0, \ldots, y_{p-1}$ are positive integers. The proof is similar to the proof of Theorem 9. Given $x \in \mathbb{P}$, we consider the extended board B_x by adding x-levels of length n below B. Then suppose that $y_0, \ldots, y_{p-1}$ are fixed elements of $\mathbb{P}$. For a given

$P \in \mathcal{N}_n^p(B_x)$, we let

$$wt(P) = \left(\prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P \cap B)}\right) q^{\mathrm{inv}(P) + \sum_{j=0}^{p-1}(y_j - 1)E_j(P \cap B)}.$$

Then we claim that (41) arises by calculating

$$S(B, q, y_0, \ldots, y_{p-1}) := \sum_{P \in \mathcal{N}_n^p(B_x)} wt(P)$$

in two different ways. First, we can fix a p-rook placement $Q \in \mathcal{N}_{n-k}^p(B)$ of $n-k$ rooks in B, and place k rooks below the bar. In this case, there are k columns below the bar which do not contain rooks in Q. First consider the contribution that comes from placing rooks below the bar in the first available column, reading from left to right.

If we place a rook in the top cell of the first available column, then it would contribute q^0 to $\mathrm{inv}(P)$. If we place that rook one cell below, then it would give q^1, and so on. Thus, we gain

$$q^0 + \cdots + q^{px-1} = [px]_q$$

to $\mathrm{inv}(P)$. Once we place a rook in the first available column, then we can use the same argument to show that the next available would give a contribution of $[px-p]_q$ to $\mathrm{inv}(P)$. Continuing this way, it is easy to see that we contribute

$$[px]_q [px-p]_q \cdots [px - p(k-1)]_q$$

to $\mathrm{inv}(P)$ as we extend Q to a placement in $\mathcal{N}_n^p(B_x)$. Thus

$$\begin{aligned}
S(B, q, y_0, \ldots, y_{p-1}) &= \sum_{k=0}^{n} \sum_{Q \in \mathcal{N}_{n-k}^p(B)} wt(Q) \\
&= \sum_{k=0}^{n} [px]_q \downarrow_{k,p} \sum_{Q \in \mathcal{N}_{n-k}^p(B)} \left(\prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(Q)}\right) q^{\mathrm{inv}(P) + \sum_{j=0}^{p-1}(y_j - 1)E_j(Q)} \\
&= \sum_{k=0}^{n} r_{n-k}^p(B, q, y_0, \ldots, y_{p-1}) [px]_q \downarrow_{k,p}
\end{aligned}$$

which is the left hand side of (41).

On the other hand, we can calculate $S(B, q, y_0, \ldots, y_{p-1})$ by adding rooks column by column, reading from left to right. We need the following analogue of Lemma 10.

Lemma 15. *Suppose that $Q \in \mathcal{N}_t^p(B_x)$ is a p-rook placement of t rooks in the first $i-1$ columns of B_x. Let $D_i(Q)$ denote the set of all p-rook placements P*

that result from Q by adding a rook in column i. Then

$$\sum_{P\in D_i(Q)} wt(P) = \begin{cases} [b_i + px - p(t+1) + y_0 + \cdots + y_{p-1}]_q wt(Q) & \textit{if } b_i \geq pi, \\ [b_i + px - p(t+1) + p]_q wt(Q) & \textit{if } b_i < pi. \end{cases} \tag{42}$$

Proof. The proof is similar to the proof of Lemma 10. That is, if $b_i < pi$, then any placement of a rook in column i will not contribute to $E_j(P\cap B)$ for any j. Now there are $px + b_i - p(i-1)$ uncancelled squares in the i-th column. If we place a rook r_i in the j-th uncancelled cell from the top in column i, then r_i will contribute a factor q^{j-1} to $wt(P)$ as the contribution to $inv(P)$ from r_i will be $j-1$. Thus in this case, the placement of r_i will contribute a factor of $wt(Q)\sum_{j=1}^{px+b_i-p(i-1)} q^{j-1} = wt(Q)[px + b_i - p(i-1)]_q$ to $\sum_{P\in D_i(Q)} wt(P)$.

If $b_i \geq i$, then there is a level $l_i \leq i$ such that placing a rook r_i in level ℓ_i in column i will complete a cycle relative to the rooks in Q. Assume that if we place a rook in cell (l_i, i, s), then we complete a cycle of sign ω^{u_s}. Thus $\omega^{u_0}, \ldots, \omega^{u_{p-1}}$ must be a rearrangement of $1, \omega, \ldots, \omega^{p-1}$. In addition, assume that there are pt_i uncancelled cells above level ℓ_i in column i. Then as before, placing a rook in the j-th uncancelled cell from the top where $j \leq pt_i$, will give a factor of q^{j-1} to $\sum_{P\in D_i(Q)} wt(P)$. Thus the placements of a rook in the top pt_i cells will give a factor of $wt(Q)(1 + q + \cdots + q^{pt_i-1}) = wt(Q)[pt_i]_q$ to $\sum_{P\in D_i(Q)} wt(P)$.

Now consider the effect of placing a rook r_i in cell $(l_i, i, p-1)$. Then r_i would contribute a factor of

$$[y_{u_{p-1}}]_q q^{pt_i} = q^{pt_i} + \cdots + q^{pt_i + y_{u_{p-1}} - 1}$$

to $wt(P)$. Here $[y_{u_{p-1}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-1}}$ and q^{pt_i} comes from the contribution of r_i to $inv(P)$. Note r_i makes no contribution to $E_j(P)$ for any j in this case. Next consider the effect of placing a rook r_i in cell $(l_i, i, p-2)$. Then r_i would contribute a factor of

$$[y_{u_{p-2}}]_q q^{pt_i+1} q^{y_{u_{p-1}}-1} = q^{pt_i + y_{u_{p-1}}} + \cdots + q^{pt_i + y_{u_{p-1}} + y_{u_{p-2}} - 1}$$

to $wt(P)$. Here $[y_{u_{p-2}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-2}}$, q^{pt_i+1} comes from the contribution of r_i to $inv(P)$, and $q^{y_{u_{p-1}}-1}$ comes from the fact that the placement of r_i contributes 1 to $E_{u_{p-1}}(P)$. Next consider the effect of placing a rook r_i in cell $(l_i, i, p-3)$. Then r_i would contribute a factor of

$$\begin{aligned} &[y_{u_{p-3}}]_q q^{pt_i+2} q^{y_{u_{p-1}}-1+y_{u_{p-2}}-1} \\ &= q^{pt_i + y_{u_{p-1}} + y_{u_{p-2}}} + \cdots + q^{pt_i + y_{u_{p-1}} + y_{u_{p-2}} + y_{u_{p-3}} - 1} \end{aligned}$$

to $wt(P)$. Here $[y_{u_{p-3}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-3}}$, q^{pt_i+2} comes from the contribution of r_i to $inv(P)$, and $q^{y_{u_{p-1}}-1+y_{u_{p-2}}-1}$ comes from the fact that the placement of r_i contributes 1 to both $E_{u_{p-1}}(P)$

and $E_{u_{p-1}}(P)$. Continuing on in this way, one can show that the contribution of all the possible placements of r_i in level ℓ_i in column i contribute a factor of $wt(Q)q^{pt_i}[y_0 + \cdots y_{p-1}]_q$ to $\sum_{P \in D_i(Q)} wt(P)$.

We have $px + b_i - p(i-1) - pt_i - p$ uncancelled cells below level ℓ_i in column i. If we place a rook r_i in the s-th such cell reading from the top, then r_i contributes a factor of $q^{pt_i+p+s-1}q^{\sum_{j=0}^{p-1} y_i - 1} = q^{pt_i+y_0+\cdots+y_{p-1}+s-1}$ to $wt(P)$. Here $q^{pt_i+p+s-1}$ comes from the r_i contribution to $inv(P)$ and $q^{\sum_{j=0}^{p-1} y_i - 1}$ comes from the fact that r_i would contribute 1 to $E_j(P)$ for $j = 0, \ldots, p-1$. It follows that the contribution to $\sum_{P \in D_i(Q)} wt(P)$ over all possible placements of rooks in the remaining $px + b_i - p(i-1) - pt_i - p$ uncancelled cells is

$$wt(Q)q^{pt_i+y_0+\cdots+y_{p-1}}[px + b_i - p(i-1) - pt_i - p]_q.$$

Hence the total contribution to $\sum_{P \in D_i(Q)} wt(P)$ of the placements of rooks in the i-th column in the case where $b_i \geq pi$ is

$$\begin{aligned} &wt(Q)([pt_i]_q + q^{pt_i}[\sum_{i=0}^{p-1} y_i]_q + q^{pt_i+\sum_{i=0}^{p-1} y_i}[px + b_i - p(i-1) - pt_i - p]_q) \\ &= wt(Q)[px + b_i - pi + y_0 + \cdots + y_{p-1}]_q \end{aligned}$$

as desired. □

If we start with a placement $Q \in \mathcal{N}_{i-1}^p(B_x)$ of $i-1$ rooks in the first $i-1$ columns of B_x, then we get the factor $[px + b_i - pi + y_0 + \cdots + y_{p-1}]_q$ from placing a rook in the column i if $b_i \geq pi$ and the factor $[px + b_i - p(i-1)]_q$ if $b_i < pi$. Thus,

$$\begin{aligned} &S(B, q, y_0, \ldots, y_{p-1}) = \\ &\qquad \prod_{i:b_i<pi} [px + b_i - p(i-1)]_q \prod_{i:b_i \geq pi} [px + b_i - pi + y_0 + \cdots + y_{p-1}]_q \end{aligned}$$

which is the right hand side of (41). □

We say that full Ferrers board $B = F(b_1, \ldots, b_n) \subseteq B_p^n$ is *regular* if $b_i = p \cdot c_i$ where $c_i \geq i$ for $1 \leq i \leq n$, For the rest of the paper, we will only consider the regular full Ferrers board $B = F(b_1, \ldots, b_n) \subseteq B_n^p$. Now if $B = F(pc_1, \ldots, pc_n)$ is a regular full Ferrers board contained in B_n^p, then, in the notation $B = B(h_1^p, d_1; \ldots; h_t^p, d_t)$, $h_j^p = p \cdot h_j$ where h_j's are the number of levels of the corresponding step. Then by Theorem 14,

$$\sum_{k=0}^{n} r_{n-k}^p(B, q, y_0, \ldots, y_{p-1})[px]_q \downarrow_{k,p} = \prod_{i=1}^{n} [px + p(c_i - i) + y_0 + \cdots + y_{p-1}]_q. \quad (43)$$

We let the right hand side of (43) be

$$\text{PR}[B, x, y_0, \ldots, y_{p-1}] := \prod_{i=1}^{n} [px + p(c_i - i) + y_0 + \cdots + y_{p-1}]_q.$$

Define $A^p_{n,k}(B,q,y_0,\ldots,y_{p-1})$ by

$$\sum_{k=0}^{n} r^p_{n-k}(B,q,y_0,\ldots,y_{p-1})[y_0+\cdots+y_{p-1}]_q\uparrow_{k,p} z^k \times \prod_{i=k+1}^{n}(1-zq^{y_0+\cdots+y_{p-1}+p(i-1)}) := \sum_{k=0}^{n} A^p_{n,k}(B,q,y_0,\ldots,y_{p-1})z^k. \quad (44)$$

Note that when $q=1$, by changing z to z^{-1} and multiplying z^n on both sides, we can transform (44) to

$$\sum_{k=0}^{n} A^p_{n,n-k}(B,1,y_0,\ldots,y_{p-1})z^k = \sum_{k=0}^{n} r^p_k(B,1,y_0,\ldots,y_{p-1})(y_0+\cdots+y_{p-1})\uparrow_{n-k,p}(z-1)^k.$$

By comparing it to the result of Theorem 7, we can see that

$$A^p_{n,k}(B,1,y_0,\ldots,y_{p-1}) = H^p_{n-k,n}(B,y_0,\ldots,y_{p-1}).$$

To derive a recursion of $A^p_{n,k}(B,q,y_0,\ldots,y_{p-1})$, we define a more general version of it. That is, we define $A^p_{n,k}(B,x,q,y_0,\ldots,y_{p-1})$ by

$$\sum_{k=0}^{n} A^p_{n,k}(B,x,q,y_0,\ldots,y_{p-1})z^k := \sum_{k=0}^{n} r^p_{n-k}(B,q,y_0,\ldots,y_{p-1})[px]_q\uparrow_{k,p} z^k \times \prod_{i=k+1}^{n}(1-zq^{px+p(i-1)}).$$

Remark 1. *We note that*

$$A^p_{n,k}(B,q,y_0,\ldots,y_{p-1}) = A^p_{n,k}(B,x,q,y_0,\ldots,y_{p-1})\Big|_{x=\frac{y_0+\cdots+y_{p-1}}{p}},$$

and $A^p_{n,k}(B,q,y_0,\ldots,y_{p-1})$ is a generalization of $A_k(x,y,B)$ as defined by Haglund in [14] and used by Butler in [5].

The following two propositions are generalizations of results of Haglund [14, Lemma 5.1, Lemma 5.7].

Proposition 1. *Suppose $B=F(pb_1,\ldots,pb_n)$ is a regular full Ferrers board contained in B^p_n. Then we have*

$$A^p_{n,k}(B,x,y_0,\ldots,y_{p-1}) = \sum_{j=0}^{k}\begin{bmatrix} n+x \\ k-j \end{bmatrix}_{q^p}\begin{bmatrix} x+j-1 \\ j \end{bmatrix}_{q^p}(-1)^{k-j}q^{p\binom{k-j}{2}}PR[B,j,y_0,\ldots,y_{p-1}], \quad (45)$$

where $PR[B,j,y_0,\ldots,y_{p-1}] = \prod_{i=1}^{n}[pj+p(c_i-i)+y_0+\cdots+y_{p-1}]_q$.

Proof. The right hand side of (45) is

$$\sum_{j=0}^{k} \left[\begin{array}{c} n+x \\ k-j \end{array} \right]_{q^p} (-1)^{k-j} q^{p\binom{k-j}{2}} \left[\begin{array}{c} x+j-1 \\ j \end{array} \right]_{q^p} \times \sum_{s=0}^{n} [p]^s [j]_{q^p} [j-1]_{q^p} \cdots [j-s+1]_{q^p} r_{n-s}^p (B, q, y_0, \ldots, y_{p-1})$$

$$= \sum_{s=0}^{n} [p]^s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \times \sum_{j \geq s} \left[\begin{array}{c} n+x \\ k-j \end{array} \right]_{q^p} (-1)^{k-j} q^{p\binom{k-j}{2}} \frac{([x]_{q^p})_j}{([1]_{q^p})_j} [j]_{q^p} \cdots [j-s+1]_{q^p}$$

$$= \sum_{s=0}^{n} [p]^s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \times \sum_{\substack{u \geq 0 \\ j=u+s}} \left[\begin{array}{c} n+x \\ k-u-s \end{array} \right]_{q^p} (-1)^{k-u-s} q^{p\binom{k-u-s}{2}} \frac{([x]_{q^p})_{u+s}}{([1]_{q^p})_{u+s}} ([u+1]_{q^p})_s$$

$$= \sum_{s=0}^{n} [p]^s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \sum_{u \geq 0} \left[\begin{array}{c} n+x \\ k-s \end{array} \right]_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} \times \frac{([s-k]_{q^p})_u}{([n+x-k+s+1]_{q^p})_u} \frac{([x]_{q^p})_s ([x+s]_{q^p})_u}{([s+1]_{q^p})_u} \left[\begin{array}{c} u+s \\ u \end{array} \right]_{q^p}$$

$$= \sum_{s=0}^{n} [p]^s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \sum_{u \geq 0} \left[\begin{array}{c} n+x \\ k-s \end{array} \right]_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} ([x]_{q^p})_s \times \sum_{u \geq 0} \frac{([-k+s]_{q^p})_s ([x+s]_{q^p})_u}{([1]_{q^p})_u ([n+x-k+s+1]_{q^p})_u}$$

$$= \sum_{s=0}^{n} [p]^s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \times \left[\begin{array}{c} n+x \\ k-s \end{array} \right]_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} ([x]_{q^p})_s \frac{([n-k+1]_{q^p})_{k-s}}{([n+x-k+s+1]_{q^p})_{k-s}}$$

$$= \sum_{s=0}^{n} [p]^s ([x]_{q^p})_s r_{n-s}^p (B, q, y_0, \ldots, y_{p-1}) \left[\begin{array}{c} n-s \\ k-s \end{array} \right]_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}}$$

$$= A_{n,k}^p (B, x, y_0, \ldots, y_{p-1}).$$

□

Proposition 2. *Suppose $B = F(pb_1, \ldots, pb_n)$ is a regular full Ferrers board contained in B_n^p. Let $H_i := h_1 + \cdots + h_i$, $D_i := d_1 + \cdots + d_i$, and the notation $B - h_i - d_j$ refers to the board obtained from B by decreasing h_i and d_j by* *one each and leaving the other parameters fixed. Then we have the following recursion for* $A^p_{n,k}(B,x,y_0,\ldots,y_{p-1})$.

$$A^p_{n,k}(B,x,y_0,\ldots,y_{p-1}) =$$
$$[p]\left[k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p} A^p_{n-1,k}(B-h_l-d_l,x,y_0,\ldots,y_{p-1})$$
$$+[p]\Bigg(-q^{p(n+x-1)}\left[k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}$$
$$+q^{p(k+H_l-D_l+d_l-2+\frac{y_0+\cdots+y_{p-1}}{p})}[n+x]_{q^p}\Bigg)\times A^p_{n-1,k-1}(B-h_l-d_l,x,y_0,\ldots,y_{p-1}).$$

Proof.

$$A^p_{n,k}(B,x,y_0,\ldots,y_{p-1})$$
$$=\sum_{s=0}^{k}\begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p}\begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p}(-1)^{k-s}q^{p\binom{k-s}{2}}\prod_{i=1}^{n}[ps+p(c_i-i)+y_0+\cdots+y_{p-1}]_q$$
$$=\sum_{s=0}^{k}\begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p}\begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p}(-1)^{k-s}q^{p\binom{k-s}{2}}$$
$$\times[ps+p(H_l-D_l+d_l-1)+y_0+\cdots+y_{p-1}]_q\mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]$$

$$=\sum_{s=0}^{k}\begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p}\begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p}(-1)^{k-s}q^{p\binom{k-s}{2}}\mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]$$
$$\times\Bigg\{[p]\left[k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}$$
$$-q^{p(s+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p})}[p][k-s]_{q^p}\Bigg\}$$
$$=[p]\left[k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}$$
$$\times\sum_{s=0}^{k}\begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p}\begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p}(-1)^{k-s}q^{p\binom{k-s}{2}}\mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]$$
$$-[p]q^{p(s+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p})}\times\sum_{s=0}^{k}[n+x]_{q^p}\begin{bmatrix} n+x-1 \\ k-s-1 \end{bmatrix}_{q^p}\begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p}$$
$$\times(-1)^{k-s}q^{p(\binom{k-s}{2}+s)}\mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]$$

$$
\begin{aligned}
&= [p]\left[k + H_l - D_l + d_l - 1 + \frac{y_0 + \cdots + y_{p-1}}{p}\right]_{q^p} \sum_{s=0}^{k} \left[\begin{array}{c} x+s-1 \\ s \end{array}\right]_{q^p} (-1)^{k-s} \\
&\times \left\{ \left[\begin{array}{c} n+x-1 \\ k-s \end{array}\right]_{q^p} q^{p\binom{k-s}{2}} + \left[\begin{array}{c} n+x-1 \\ k-s-1 \end{array}\right]_{q^p} q^{p\left(\binom{k-s-1}{2}+n+x-1\right)} \right\} \\
&\qquad\qquad \times \mathrm{PR}[B - h_l - d_l, s, y_0, \ldots, y_{p-1}] \\
&- [p] q^{p(s + H_l - D_l + d_l - 1 + \frac{y_0+\cdots+y_{p-1}}{p})} \times \sum_{s=0}^{k-1} [n+x]_{q^p} \left[\begin{array}{c} n+x-1 \\ k-s-1 \end{array}\right]_{q^p} \left[\begin{array}{c} x+s-1 \\ s \end{array}\right]_{q^p} \\
&\qquad \times (-1)^{k-s} q^{p\left(\binom{k-s-1}{2}+k-1\right)} \mathrm{PR}[B - h_l - d_l, s, y_0, \ldots, y_{p-1}] \\
&= [p]\left[k + H_l - D_l + d_l - 1 + \frac{y_0 + \cdots + y_{p-1}}{p}\right]_{q^p} A^p_{n-1,k}(B - h_l - d_l, x, y_0, \ldots, y_{p-1}) \\
&+ [p] A^p_{n-1,k-1}(B - h_l - d_l, x, y_0, \ldots, y_{p-1}) \left\{ q^{p(k + H_l - D_l + d_l - 2 + \frac{y_0+\cdots+y_{p-1}}{p})} [n+x]_{q^p} \right. \\
&\qquad \left. - q^{p(n+x-1)} \left[k + H_l - D_l + d_l - 1 + \frac{y_0 + \cdots + y_{p-1}}{p}\right]_{q^p} \right\} \\
&= [p]\left[k + H_l - D_l + d_l - 1 + \frac{y_0 + \cdots + y_{p-1}}{p}\right]_{q^p} A^p_{n-1,k}(B - h_l - d_l, x, q, y_0, \ldots, y_{p-1}) \\
&+ [p] q^{p(k + H_l - D_l + d_l - 2 + \frac{y_0+\cdots+y_{p-1}}{p})} \left[n + x - k - H_l + D_l - d_l + 1 - \frac{y_0 + \cdots + y_{p-1}}{p}\right]_{q^p} \\
&\qquad \times A^p_{n-1,k-1}(B - h_l - d_l, x, q, y_0, \ldots, y_{p-1}).
\end{aligned}
$$

□

Proposition 3. *If $B_j = B(h^p_1, d_1; \cdots; h^p_{l-1}, d_{l-1}; h^p_l - pj, d_l - j; h^p_{l+1}, d_{l+1}; \ldots; h^p_t, d_t)$ is the board obtained from a regular Ferrers board B by decreasing h^p_l by pj and d_l by j (here we assume that $j \leq h_l, d_l$ where $h^p_l = ph_l$), then*

$$
\begin{aligned}
A^p_{n,k}(B, x, q, y_0, \ldots, y_{p-1}) &= [p]^j_q [j]_{q^p}! \sum_{s=k-j}^{k} A^p_{n,s}(B_j, x, q, y_0, \ldots, y_{p-1}) \\
&\times \left[\begin{array}{c} T_l - 1 + s \\ j - k + s \end{array}\right]_{q^p} \left[\begin{array}{c} n - T_l + x - s \\ k - s \end{array}\right]_{q^p} q^{p(k-s)(T_l+k-j-1)}, \quad (46)
\end{aligned}
$$

where $T_l = H_l - D_{l-1} + \frac{y_0+\cdots+y_{p-1}}{p}$.

Proof. The proof can be done by induction on j and by using the recursion in Proposition 2. The proof is similar to the proof of [14, Theorem 4.1, Theorem 5.8]. □

By using Proposition 2, we can derive a recursion for $A^p_{n,k}(B, q, y_0, \ldots, y_{p-1})$.

Theorem 16. *Suppose $B = F(pb_1, \dots, pb_n)$ is a regular full Ferrers board contained in B_n^p. Let $H_i := h_1 + \cdots + h_i$, $D_i := d_1 + \cdots + d_i$, and the notation $B - h_i - d_j$ refers to the board obtained from B by decreasing h_i and d_j by one each and leaving the other parameters fixed. Then we have the following recursion for $A_{n,k}^p(B, q, y_0, \dots, y_{p-1})$.*

$$\begin{aligned} &A_{n,k}^p(B, q, y_0, \dots, y_{p-1}) = \qquad (47)\\ &[p]\left[\frac{y_0 + \cdots + y_{p-1}}{p} + k + d_t - 1\right]_{q^p} A_{n-1,k}^p(B - h_t - d_t, q, y_0, \dots, y_{p-1})\\ &+ [p]q^{p\left(\frac{y_0+\cdots+y_{p-1}}{p}+k+d_t-2\right)}[n-k-d_t+1]_{q^p} A_{n-1,k-1}^p(B - h_t - d_t, q, y_0, \dots, y_{p-1}),\end{aligned}$$

where h_t and d_t are the height (number of levels) and the depth of the last step of B.

We note that it follows from Theorem 16 that if $B = F(pb_1, \dots, pb_n)$ is a regular full Ferrers board in B_n^p and $y_0, \dots, y_{p-1}$ are non-negative integers, then $A_{n,k}^p(B, q, y_0, \dots, y_{p-1})$ is a polynomial with non-negative coefficients in q. Here are some small examples.

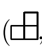
Example 1. *When B_1 has only one square, i.e., $B_1 = F(p)$, then*

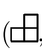
$$\begin{aligned} &A_{1,0}^p(B_1, q, y_0, \dots, y_{p-1})\\ &= r_1^p = \sum_{P \in \mathcal{N}_1^p(B_1)} \left[\prod_{j=0}^{p-1} [y_j]^{cyc_j(P)} q^{inv(P) + \sum_{j=0}^{p-1}(y_j - 1)E_j(P)}\right]\\ &= [y_{p-1}] \cdot q^0 + [y_{p-2}] \cdot q^{1+(y_{p-1}-1)} + \cdots + [y_0] \cdot q^{p-1+\sum_{j=1}^{p-1}(y_j-1)}\\ &= [y_{p-1}] \cdot q^0 + [y_{p-2}] \cdot q^{y_{p-1}} + [y_{p-3}] \cdot q^{y_{p-1}+y_{p-2}} + \cdots + [y_0] \cdot q^{\sum_{j=1}^{p-1} y_j}\\ &= [y_0 + \cdots + y_{p-1}],\end{aligned}$$

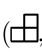
$A_{1,k}^p(B_1, q, y_0, \dots, y_{p-1}) = 0,$ *for* $k > 0$.

$A_{2,0}^p($$, q, y_0, \dots, y_{p-1}) = [y_0 + \cdots + y_{p-1}][y_0 + \cdots + y_{p-1} + p],$

$A_{2,k}^p($$, q, y_0, \dots, y_{p-1}) = 0,$ *for* $k > 0$.

$A_{2,0}^p($$, q, y_0, \dots, y_{p-1}) = [y_0 + \cdots + y_{p-1}]^2,$

$A_{2,1}^p($$, q, y_0, \dots, y_{p-1}) = q^{(y_0+\cdots+y_{p-1})}[p][y_0 + \cdots + y_{p-1}],$

$A_{2,2}^p($$, q, y_0, \dots, y_{p-1}) = 0.$

Based on the q-statistics for the cycle counting hit numbers defined by Butler in [5], we conjecture similar q-statistics for the cycle counting p-hit numbers. Before we make a precise statement, we need some definitions.

For a full regular Ferrers board $B \subseteq B_n^p$, let $\mathcal{N}^p(B) = \cup_{k=1}^n \mathcal{N}_k^p(B)$. For $p \in \mathcal{N}^p(B)$, Butler's statistic $s_{B,b}(P)$ [5] is defined as the number of squares on B_n^p which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme :

1. Each rook cancels all squares to the right in its row,
2. each rook on B cancels all squares above it in its column (squares both on B and strictly above B),
3. each rook on B which also completes a cycle cancels all squares below it in its column as well,
4. each rook off B cancels all squares below it but above B.

Define $\mathrm{cyc}_{\geq j}(P)$ by

$$\mathrm{cyc}_{\geq j}(P) := \sum_{i=j}^{p-1} \mathrm{cyc}_i(P).$$

Since $b_i \geq pi$, there exists a unique level, say u, in column i such that, considering only rooks from P in column 1 through column $i-1$ of B, completes a cycle. At i^{th} column, define $\tilde{E}_i(P)$ by

$$\tilde{E}_i(P) = \begin{cases} p, & \text{if there is no rook from } P \text{ in column } i \text{ on or above the level } u \\ 0, & \text{if there is a rook from } P \text{ in column } i \text{ above the level } u \\ p-1-j, & \text{if there is a rook on the level } u \text{ completing a cycle of sign } \omega_j. \end{cases}$$

Then we conjecture the following combinatorial formula for $A_{n,k}^p(B, q, y_0, \ldots, y_{p-1})$.

Conjecture 1. *Let $\mathcal{H}_{n,k}(B)$ be the set of all placements corresponding $\sigma \in C_p \wr S_n$ such that $|\sigma \cap B| = k$. Then, for a full regular Ferrers board $B \subseteq B_n^p$,*

$$\begin{aligned} &A_{n,k}^p(B, q, y_0, \ldots, y_{p-1}) \\ &= \sum_{P \in \mathcal{H}_{n,n-k}(B)} \left(\prod_{j=0}^{p-1} [y_j]^{cyc_j(P)} \right) q^{s_{B,b}(P) + \sum_{i=1}^n \tilde{E}_i(P) + \sum_{j=0}^{p-1} [(y_j - 1)(n - cyc_{\geq j}(P))]}. \end{aligned} \tag{48}$$

An obvious approach to prove Conjecture 1 is to give a combinatorial proof that the recursion for $A_{n,k}^p(B, q, y_0, \ldots, y_{p-1})$ in (47) holds. We were not able to find a natural way to partition the rook placement in $\mathcal{N}_k(B)$ to account for the two terms on the right hand side of (47). Our next example will show that while we can verify the recursion holds for $B = F(p, 2p, 3p, 4p) \subset [4] \times [4p]$, the way that we partition the rook placements in B to account for the two terms on the right hand side of (47) is quite complicated. Thus we do not see how the recursion can be derived naturally by extending the rook placements corresponding to the permutations in S_{n-1}.

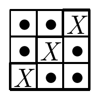
Example 2. *We consider a staircase board $B = F(p, 2p, 3p, 4p) \subset [4] \times [4p]$. Then $B - h_4 - d_4 = F(p, 2p, 3p)$ and the recursion (47) when $k = 1$ is*

$$A^p_{4,1}(B, q, y_0, \ldots, y_{p-1}) = [y_0 + \cdots + y_{p-1} + p]_q A^p_{3,1}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}) \\ + q^{y_0 + \cdots + y_{p-1}}[p]_q[3]_{q^p} A^p_{3,0}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}). \quad (49)$$

For a rook placement $P \in \mathcal{H}_{n,n-k}(B)$, let

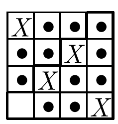
$$wt(P) = \left(\prod_{j=0}^{p-1}[y_j]^{cyc_j(P)}\right) q^{s_{B,b}(P) + \sum_{i=1}^{n} \tilde{E}_i(P) + \sum_{j=0}^{p-1}[(y_j - 1)(n - cyc_{\geq j}(P))]}.$$

Then for $\sigma = (1)(2)(3) \in S_3$,

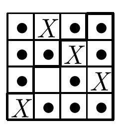


$$A^p_{3,0}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}) = \sum_{P \in C_p \wr \sigma} wt(P) = [y_0 + \cdots + y_{p-1}]^3_q.$$

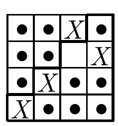
This can be extended to a placement in $\mathcal{H}_{4,3}(B)$ as follows.



$$\sigma_1 = (14)(2)(3), \sum_{P \in C_p \wr \sigma_1} wt(P) = [y_0 + \cdots + y_{p-1}]^3_q q^{y_0 + \cdots + y_{p-1}}[p]_q, \quad (50)$$

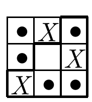


$$\sigma_2 = (1)(24)(3), \sum_{P \in C_p \wr \sigma_2} wt(P) = [y_0 + \cdots + y_{p-1}]^3_q q^{y_0 + \cdots + y_{p-1}}[p]_q, \quad (51)$$



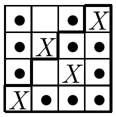
$$\sigma_3 = (1)(2)(34), \sum_{P \in C_p \wr \sigma_3} wt(P) = [y_0 + \cdots + y_{p-1}]^3_q q^{y_0 + \cdots + y_{p-1}}[p]_q. \quad (52)$$

There are four permutations in S_3 which contribute to $A^p_{3,1}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$ and they can be extended to a placement in $\mathcal{H}_{4,3}$ as follows.

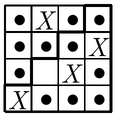


$$\alpha = (1)(23), \sum_{C_p \wr \alpha} wt(P) = [y_0 + \cdots + y_{p-1}]^2_q q^{y_0 + \cdots + y_{p-1}}[p]_q$$

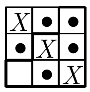
$\Rightarrow$



$$\alpha_1 = (1)(23)(4), \sum_{C_p \wr \alpha_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}+p}[p]_q, \tag{53}$$

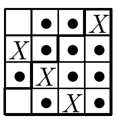


$$\alpha_2 = (1)(243), \sum_{C_p \wr \alpha_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})}[p]_q^2, \tag{54}$$

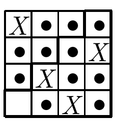


$$\beta = (13)(2), \sum_{C_p \wr \beta} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p]_q$$

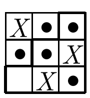
$\Rightarrow$



$$\beta_1 = (13)(2)(4), \sum_{C_p \wr \beta_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}+p}[p]_q, \tag{55}$$

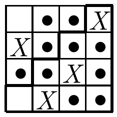


$$\beta_2 = (143)(2), \sum_{C_p \wr \beta_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})}[p]_q^2, \tag{56}$$

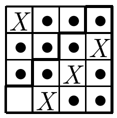


$$\gamma = (132), \sum_{C_p \wr \gamma} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})}[p]_q^2$$

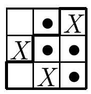
$\Rightarrow$



$$\gamma_1 = (132)(4), \sum_{C_p \wr \gamma_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})+p} [p]_q^2, \tag{57}$$

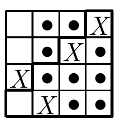


$$\gamma_2 = (1432), \sum_{C_p \wr \gamma_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q q^{3(y_0+\cdots+y_{p-1})} [p]_q^3, \tag{58}$$

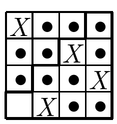


$$\delta = (12)(3), \sum_{C_p \wr \delta} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}+p} [p]_q$$

$\Rightarrow$



$$\delta_1 = (12)(3)(4), \sum_{C_p \wr \delta_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}+2p} [p]_q, \tag{59}$$



$$\delta_2 = (142)(3), \sum_{C_p \wr \delta_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})} [p]_q^2. \tag{60}$$

(50)+(55)+(59) *has a common factor* $q^{y_0+\cdots+y_{p-1}}[p]_q[3]_{q^p}$ *which is the coefficient of* $A^p_{3,0}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$ *in (49) and the rest makes* $A^p_{3,0}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$.

$$\begin{aligned}(50) + (55) + (59) &= [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}} [p]_q (1 + q^p + q^{2p}) \\ &= q^{y_0+\cdots+y_{p-1}} [y_0 + \cdots + y_{p-1}]_q^3 [p]_q [3]_{q^p} \\ &= q^{y_0+\cdots+y_{p-1}} [p]_q [3]_{q^p} A^p_{3,0}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}),\end{aligned}$$

Similarly, ((51) + (60))*,* ((52) + (56))*,* ((54) + (58)) *and* ((53) + (57)) *have a common factor* $[y_0 + \cdots + y_{p-1} + p]_q$ *which is the coefficient of* $A^p_{3,1}(B - h_4 -$

$d_4, q, y_0, \ldots, y_{p-1})$ *in (49).*

$$
\begin{aligned}
&((51)+(60))+((52)+(56))+((54)+(58))+((53)+(57))\\
&= \big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p]_q([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p]_q)\big)\\
&\quad+\big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p]_q([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p]_q)\big)\\
&\quad+\Big([y_0+\cdots+y_{p-1}]_q q^{2(y_0+\cdots+y_{p-1})}[p]_q^2([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p]_q)\Big)\\
&\quad+\big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}+p}[p]_q([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p]_q)\big)\\
&= ([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p]_q)\\
&\qquad\times\big\{[y_0+\cdots+y_{p-1}]_q q^{y_0+\cdots+y_{p-1}}[p]([y_0+\cdots+y_{p-1}]_q+q^{y_0+\cdots+y_{p-1}}[p])\\
&\qquad\qquad+[y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p](1+q^p)\big\}\\
&= [y_0+\cdots+y_{p-1}+p]_q\\
&\quad\times q^{y_0+\cdots+y_{p-1}}[p]_q[y_0+\cdots+y_{p-1}]_q\left([y_0+\cdots+y_{p-1}+p]_q+[2]_q[y_0+\cdots+y_{p-1}]_q\right)\\
&= [y_0+\cdots+y_{p-1}+p]_q A_{3,1}^p(B-h_4-d_4,q,y_0,\ldots,y_{p-1}).
\end{aligned}
$$

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