# ROOK THEORY AND HYPERGEOMETRIC SERIES

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Abstract. The number of ways of placing $k$ non-attacking rooks on a Ferrers board is expressed as a hypergeometric series, of a type originally studied by Karlsson and Minton. Known transformation identities for series of this type translate into new theorems about rook polynomials.

## 1. Introduction.

Since its introduction in the 1940's by Riordan and Kaplansky, *rook theory* has continued to find application to an ever-expanding list of topics in Enumerative Combinatorics. In this article we establish a connection between rook polynomials and certain types of *hypergeometric series*, and explore the consequences. *Notation* : LHS and RHS are abbreviations for "left hand side" and "right hand side" respectively. $\mathbb{N}$ = the nonnegative integers, $\mathbb{Z}$ = the integers, $\mathbb{C}$ = the complex numbers, "COEF($z^k$) in" means "the coefficient of $z^k$ in".

Consider an infinite grid of squares, with the same labelling as the points in the first quadrant having positive integral coordinates; the lower left-hand square has (column,row) coordinates (1,1), etc.. A *board* $B$ is a finite subset of these squares, together with a value of $n$, called the number of columns. The squares of $B$ must satisfy $(i,j) \in B \implies 1 \le i \le n,\ 1 \le j$. If in addition $(i,j) \in B \implies j \le n$ (all the squares of $B$ are contained in the $n \times n$ grid) then $B$ is called *admissible*. See Figure 1.

Let $r_k(B)$ be the number of ways of placing $k$ rooks on the squares of $B$ (throughout the article, all placements are assumed to be non-attacking, i.e. no two rooks in the same row, and no two in the same column). If $B$ is admissible, let $a_k(B)$ be the number of ways of placing $n$ non-attacking rooks on the square $n \times n$ grid with exactly $n-k$ rooks on $B$. The $a_k$ are usually called "hit" numbers. Of particular interest is $a_n$, which equals the number of permutations on $n$ letters which avoid the "forbidden" positions encoded by the squares of $B$ (we can identify a rook on square $(i,j)$ with the condition that $i$ is sent to $j$ in the associated permutation). The $a_k(B)$ can be expressed in terms of the $r_k(B)$ via an identity of Riordan and Kaplansky [KaRi]

$$\sum_k k! r_{n-k}(B)(z-1)^{(n-k)} = \sum_k z^k a_{n-k}(B). \tag{1}$$

If $B$ is not admissible, define $a_k(B)$ via (1) (although they no longer count permutations).

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FIGURE 1. The shaded squares (1,2), (2,1), and (3,3) of the $3 \times 3$ grid form an admissible board $B$.

A *Ferrers board* $B$ is a board with the property that $(i, j) \in B$ implies all squares to the right and below $(i, j)$ are also in $B$. More formally, $(i, j) \in B \implies (k, p) \in B$ for $i \le k \le n$ and $1 \le p \le j$ . These boards can be identified with the Ferrers graphs of partitions. They were introduced by Foata and Schützenberger, who proved that every Ferrers board is *rook equivalent* (has the same rook numbers $r_k$) to a unique board with strictly increasing column heights. Ferrers boards satisfy the important *factorization theorem* of Goldman, Joichi, and White [GJW1]

$$\sum_{k=0}^{n} x(x-1)\cdots(x-k+1)r_{n-k} = PR(x, B), \tag{2}$$

where $PR(x, B) = \prod_{i=1}^{n}(x + c_i - i + 1)$, with $c_i$ = the height of the $i^{th}$ column of $B$.

Throughout this article, if $B$ is a Ferrers board it will represent the board of Figure 2, indicated by the following notation: $B = B(h_1, d_1; h_2, d_2; \dots ; h_t, d_t)$. In order to allow leading columns of height zero and for other technical reasons we allow the $h_i$ to be nonnegative integers, but the $d_i$ will be strictly positive integers. Note that $PR(x, B)$ can be written as

$$\prod_{i=1}^{t}(x + H_i - D_i + 1)_{d_i}, \tag{3}$$

where $H_i := h_1 + \ldots + h_i$, $D_i := d_1 + d_2 + \ldots + d_i$ (this notation will be used often), and $(x)_k := x(x+1)\cdots(x+k-1)$.

For some time researchers have sought a $q$-version of (1), the inclusion-exclusion identity of Riordan and Kaplansky. For arbitrary boards this problem has never been completely solved, although partial solutions occur in [ChRo] and [JoRo]. For Ferrers boards Garsia and Remmel [GaRe] introduced a $q$-version which has found a number of applications [Din1], [Din2], [Hag1]. In particular, Solomon [Sol] has developed connections between the monoid of matrices over a finite field

FIGURE 2. The Ferrers board $B = B(h_1, d_1; \ldots ; h_t, d_t)$.

and $q$-rook polynomials, and Ding has unearthed an exciting connection between algebraic topology and rook placements by showing that the Poincaré polynomials of cohomolgy for certain algebraic varieties are expressable as $q$-rook polynomials.

Other recent work in rook theory incorporates the cycle structure of *simple directed graphs* associated to rook placements. This idea originated in a 1989 paper of Gessel [Ges1]; if a rook occupies square $(i, j)$, draw an edge from $i$ to $j$ in the associated digraph (otherwise do not draw such an edge). The resulting digraph (on $n$ vertices) will consist of a certain number of cycles and a certain number of directed paths (vertices with no incident edges count as a directed path of length one). See Figure 3.

Let

$$r_k(y) := \sum_{\text{placements of } k \text{ rooks on } B} y^{\text{number of cycles}},$$

so for the placement of Figure 3 we associate $y^2$. If $B$ is admissible, we can define

$$a_k(y, B) := \sum_{\substack{\text{placements of } n \text{ rooks on } n\times n \text{ square} \\ n-k \text{ rooks on } B}} y^{\text{number of cycles}}. \tag{4}$$

It should be mentioned that the special problem of determining $a_0(B)$, which can be viewed as the permanent of a matrix, has been studied in great detail by Shevelev. His work also contains some results on determining $a_0(y, B)$; see [Shev] and the long list of references it contains.

Chung and Graham introduced the function

$$C(B; x, y) := \sum_k x(x-1)\cdots(x-k+1)r_{n-k}(y). \tag{5}$$

FIGURE 3. A rook placement and the associated digraph.

One of their results can be expressed as follows [ChG]

$$C(B;x,y) = \sum_{\substack{S \\ \text{0 to } n \text{ rooks on } B \\ \text{each rook in a cycle}}} (y-1)^{\text{number of cycles of S}} \times \sum_{\substack{T \\ \text{n rooks on } n\times n \text{ square} \\ S\subseteq T}} \binom{x+|T\cap B|-|S|}{n-|S|}$$

where the inner sum is over all placements $T$ of $n$ non-taking rooks which contain the rooks in $S$, and $|T \cap B|$ is the number of rooks in $T$ on $B$. In the outer sum, each rook in $S$ must be in a cycle.

Gessel [Ges2] found a more compact expansion for $C(B;x,y)$;

$$C(B;x,y) = \sum_k a_{n-k}(y) \frac{(x+y)_k x(x-1)\cdots(x-n+k+1)}{(y)_n}. \tag{6}$$

He also noted that

$$\sum_k (y)_k r_{n-k}(y)(z-1)^{n-k} = \sum_k z^k a_{n-k}(y). \tag{7}$$

Shortly after a preprint of Chung and Graham's influential work became available, the author and Dworkin noticed independently that a version of the factorization theorem for Ferrers boards held for $r_k(y)$ [EHR], [Dwo]

$$\sum_k x(x-1)\cdots(x-k+1)r_{n-k}(y) = \prod_{c_i\geq i}(x+c_i-i+y)\prod_{c_i<i}(x+c_i-i+1). \tag{8}$$

Dworkin also investigated if and when the LHS of (8) factors for those boards obtained by permuting the columns of a Ferrers board.

Earlier Stanley and Stembridge [StS] developed a version of rook theory which takes into account the cycle structure of rook placements and the associated digraph. To describe this we need two partitions $\alpha$, $\beta$. The $\alpha_i$ are the lengths of the directed paths, and the $\beta_i$ are the lengths of the cycles. In their theory they weight a given placement by $f_\alpha(Y)p_\beta(Y)\prod_i m_i(\alpha)!$, where the $f_\alpha$ are the forgotten symmetric functions in the set of variables Y, $p_\beta$ are the power-sum symmetric functions, and $m_i(\alpha)$ is the multiplicity of $i$ in $\alpha$ (see [Mac] for background on symmetric functions).

Chow has recently considered a more general function;

$$C(B;X,Y) := \sum_{\alpha,\beta} m_\alpha(X)p_\beta(Y)r_{\alpha,\beta}(B)\prod_i m_i(\alpha)!$$

(in this article X,Y denote sets of variables and x,y complex variables). Here $r_{\alpha,\beta}(B)$ is the number of rook placements on $B$ whose digraph has directed path type $\alpha$ and cycle type $\beta$, and $m_\alpha$ is the monomial symmetric function. If X is chosen so that $\sum_i x_i^k = p_k(X) = (-1)^{k+1}p_k(Y)$, $C(B;X,Y)$ reduces to the Stanley-Stembridge function. If $p_k(X) \equiv x$, and $p_k(Y) \equiv y$, we get Chung and Graham's $C(B;x,y)$. Here we are using the well-known fact that identities involving symmetric functions can be interpreted as polynomial identities in the $p_k$.

Chow proved a "reciprocity" theorem for $C(B;X,Y)$, which says that for admissible boards $B$

$$C(B;X,Y) := \sum_{\substack{0 \text{ to } n \text{ rooks on } B^c \\ 0 \text{ rooks on } B}} f_\alpha(X,Y)p_\beta(Y)\prod_i m_i(\alpha)!(-1)^{n+\ell(\alpha)}, \tag{9}$$

where $\ell(\alpha)$ is the number of parts of $\alpha$, and $X$,$Y$ indicates the union of the two sets of variables $X$ and $Y$ (so $p_k(X,Y) = p_k(X) + p_k(Y)$). $B^c$ is the complement board consisting of those squares in the $n \times n$ grid not a part of $B$. In section 2 we introduce another parameter into Chow's function, and obtain a result which contains reciprocity and (1) as special cases. We also derive a more general form of (6), as well as an identity relating the $r_k(y,B)$ and the $r_k(y,B^c)$, which generalizes a result of Chow and Gessel.

A ${}_{t+1}F_t$ hypergeometric series is defined by

$${}_{t+1}F_t\left[\begin{matrix} c_1, & c_2, & \dots & c_{t+1} \\ & b_1, & \dots & b_t \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(c_1)_k(c_2)_k\cdots(c_{t+1})}{k!(b_1)_k(b_2)_k\cdots(b_t)_k} z^k. \tag{10}$$

If the argument $z$ is unity it will be omitted. The series in (10) converges absolutely if $|z| < 1$ or if $z = 1$ and $\Re(\sum_{i=1}^t b_i - \sum_{i=1}^{t+1} c_i) > 0$. The study of these functions goes back to Euler and Gauss. One of Gauss' famous results is

$${}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0. \tag{11}$$

If one of the numerator parameters in (10) is a negative integer, the series terminates. The terminating case of (11) is the *Vandermonde convolution*

$${}_2F_1\left[\begin{matrix} -n, & b \\ & c \end{matrix}\right] = \frac{(c-b)_n}{(c)_n}. \tag{12}$$

In section 3 we show how to express the rook numbers $r_k(y,B)$, when $B$ is the Ferrers board of Figure 2, in terms of a terminating ${}_{t+1}F_t$ with unit argument. Expressions are also derived for the hit numbers $a_k(y,B)$. These involve an important type of ${}_{t+2}F_{t+1}$ hypergeometric series known as *balanced* series, the sum of whose numerator parameters is one less then the sum of its denominator parameters (i.e. $c_1+\ldots+c_{t+2}-b_1-\ldots-b_{t+1}=-1$.

In section 4 we derive a recurrence for the $a_k$, which turns out to be equivalent to the known fact that a balanced, terminating ${}_{t+2}F_{t+1}$ can be expressed as a sum of balanced, terminating ${}_{t+1}F_t$'s. A special case of this is the famous Pfaff- Saalschütz result

$$ {}_3F_2\left[\begin{matrix}-n, & a, & b \\ & c, & a+b-c-n+1\end{matrix}\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{13}$$

Special cases of this recurrence for the $a_k$ are shown to have a purely combinatorial interpretation in terms of permutations of multisets.

For Ferrers boards, the rook polynomials appearing in (1) can themselves be expressed in terms of a ${}_{t+1}F_t$ of the following general type

$$ {}_{t+1}F_t\left[\begin{matrix}x, & b_1+d_1, & \ldots & b_t+d_t \\ & b_1, & \ldots & b_t\end{matrix};z\right],$$

where $d_i\in\mathbb{N}$. The series for the $r_k$ and the $a_k$ also have the property of the numerator parameters being a positive integer more than the corresponding denominator parameters. We will say these series are of $Karlsson-Minton$ *type*; they were first studied by Minton [Min, 1970] who proved

$$ {}_{t+2}F_{t+1}\left[\begin{matrix}w, & x, & b_1+d_1, & \ldots & b_t+d_t \\ & x+1, & b_1, & \ldots & b_t\end{matrix}\right]= $$

$$ \frac{\Gamma(x+1)\Gamma(1-w)}{\Gamma(1+x-w)}\prod_{i=1}^{t}\frac{(b_i-x)_{d_i}}{(b_i)_{d_i}}, \tag{14}$$

where $x,b_i\in\mathbb{C}$, $w\in\mathbb{Z}$, $d_i\in\mathbb{N}$, and $w\le -n$. Karlsson [Kar] showed that (14) holds for $w\in\mathbb{C}$, $\Re(w)<1-n$, and later Gasper [Gasp] found an interesting transformation which includes (14) as a special case

$$ {}_{t+2}F_{t+1}\left[\begin{matrix}w, & x, & b_1+d_1, & \ldots, & b_t+d_t \\ & x+c+1, & b_1, & \ldots, & b_t\end{matrix}\right]= $$

$$ \frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)}\prod_{i=1}^{t}\frac{(b_i-x)_{d_i}}{(b_i)_{d_i}} $$

$$ \times{}_{t+2}F_{t+1}\left[\begin{matrix}-c, & x, & 1+x-b_1, & \ldots, & 1+x-b_t \\ & x+1-w, & 1+x-b_1-d_1, & \ldots, & 1+x-b_t-d_t\end{matrix}\right], \tag{15}$$

where $w,c,x,b_i\in\mathbb{C}, d_i\in\mathbb{N}$, and $\Re(c-w)>n-1$.

When this transformation is expressed as a relation between $a_k$'s of different boards, special cases have simple combinatorial interpretations, but in general we get new identities. These are of a rather technical nature; a typical result involving generalized Stirling numbers is given in Example 3.2.

Gasper also derived two $q$-versions of (15), which involve *basic hypergeometric series*. A ${}_{t+1}\phi_t$ is defined by

$$ {}_{t+1}\phi_t \left[ \begin{matrix} x, & c_1, & \ldots, & c_t \\ & b_1, & \ldots, & b_t \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(x;q)_k (c_1;q)_k \cdots (c_t;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_t;q)_k} z^k, $$

where $|z| \leq 1$, $q$ is a real variable satisfying $0 < q < 1$, and $(w;q)_k = (1-w)(1-wq)\cdots(1-wq^{k-1})$. We will denote the infinite product $\prod_{k\geq 0}(1-wq^k)$ by $(w;q)_\infty$. If the meaning is clear from context, $(w;q)_k$ and $(w;q)_\infty$ will be abbreviated by $(w)_k$ and $(w)_\infty$, respectively. Replacing $x$ by $q^x$, $c_i, b_i$ by $q^{c_i}$ and $q^{b_i}$, and letting $q \to 1^-$, the ${}_{t+1}\phi_t$ above approaches the ${}_{t+1}F_t$ with the same arguments.

Recently Chu has derived a bilateral extension of (15) [Chu] (a bilateral series is a sum from $k = -\infty$ to $+\infty$, which can be viewed as a sum of two ${}_{t+1}F_t$'s). He also derived a $q$-version of this which contains both of Gasper's $q$-versions of (15) as special cases.

In chapters 4 and 5 of Gasper and Rahman's book [GaRa], and also in work of Sears [Sea1],[Sea2] and Slater [Sla], there are a number of expansions of the following general type

$$ \text{one } {}_{t+1}\phi_t = \text{ a sum of } t+1 \text{ other } {}_{t+1}\phi_t\text{'s}. \tag{16} $$

Although it appears to have gone unnoticed, we show how the Chu-Gasper $q$-version of (15) can be obtained by specializing one of the identities of type (16) to the Karlsson-Minton case (all of the coefficients of the ${}_{t+1}\phi_t$'s on the RHS of (16) turn out to be zero except for one). This same equation also shows how to express the series on the LHS of (14) as a finite sum of Gamma factors when the $d_i$ are allowed to be positive or negative integers. There is also an analogue of (16) for bilateral series (see [GaRa], eq. (5.4.4)) , due to Slater, which contains Chu's bilateral extension of (15) as a special case.

Not surprisingly, the $q$-rook polynomials $R_k$ of Garsia and Remmel can be expressed as basic hypergeometric series of Karlsson-Minton type (where each numerator parameter is $q^{d_i}$ times the corresponding denominator parameter). Although for the most part our results in section 5 are $q$-versions of results in previous sections, using the Heine transformation

$$ {}_2\phi_1 \left[ \begin{matrix} x, & b \\ & c \end{matrix} ; z \right] = \frac{(b)_\infty (xz)_\infty}{(c)_\infty (z)_\infty} \quad {}_2\phi_1 \left[ \begin{matrix} c/b, & z \\ & xz \end{matrix} ; b \right] \tag{17} $$

we derive some identities which have no analog in the $q = 1$ case. Actually we use the following corollary of Bowman's 1993 generalized Heine transformation [Bow1], [Bow2]

$$ {}_{t+1}\phi_t \left[ \begin{matrix} x, & c_1, & \ldots, & c_t \\ & b_1, & \ldots, & b_t \end{matrix} ; z \right] = \frac{(xz)_\infty}{(z)_\infty} \prod_{i=1}^{t} \frac{(c_i)_\infty}{(b_i)_\infty} \sum_{k=0}^{\infty} \frac{(z)_k}{(xz)_k (q)_k} h_k(\mathbf{b};\mathbf{c}), \tag{18} $$

where

$$ h_k(\mathbf{b};\mathbf{c}) := \sum_{m_1+\ldots+m_t=k} \frac{(q)_k}{(q)_{m_1}\cdots(q)_{m_t}} [c_1, b_1]_{m_1} \cdots [c_t, b_t]_{m_t}, $$

with $[w,y]_k := (w-y)(w-qy)\cdots(w-q^{k-1}y)$. When specialized to series of Karlsson-Minton type, the case $d_i \equiv 1$ of (18) reduces to a special case of a *bibasic* version of the Heine transformation (which contains two independent bases $p$ and $q$) due to Fine.

## 2. Reciprocity and the x-parameter.

We now introduce another parameter into Chow's function, and then extend his reciprocity theorem. Define

$$C(B;X,Y;z) := \sum_{\alpha,\beta} m_\alpha(X)p_\beta(Y)r_{\alpha,\beta}(B)(1-z)^{n-\ell(\alpha)}\prod m_i(\alpha)!. \qquad (19)$$

**Theorem 2.1** Let $B$ be an admissible board. Then

$$C(B;X,Y;z) = (-1)^n \sum_k z^k C_k(B;X,Y),$$

where

$$C_k(B;X,Y) = \sum_{\substack{0 \text{ to } n-k \text{ rooks on } B^c \\ k \text{ rooks on } B}} f_\alpha(X,Y)p_\beta(Y)\prod_i m_i(\alpha)!(-1)^{\ell(\alpha)}.$$

Note that if $z=0$ this reduces to (9).
*Proof* : Our proof closely follows Chow's proof of (9) [Cho,pp.7-8]. For a placement $\kappa$ of 0 to $n$ rooks on the $n\times n$ grid, let $T_\kappa$ = the set of rooks of $\kappa$ on $B$, and $E_\kappa$ = the set of rooks of $\kappa$ on $B^c$. By definition,

$$(-1)^n C_{n-k}(B;X,Y) = \sum_{\substack{\kappa \\ E_\kappa=\emptyset}} m_{\alpha(\kappa)}(X)p_{\beta(\kappa)}(Y)\prod m_i(\alpha)!\binom{n-\ell(\alpha)}{n-k}(-1)^{n-k}$$

$$= \sum_{\substack{\kappa \\ E_\kappa=\emptyset}} m_{\alpha(\kappa)}(X)p_{\beta(\kappa)}(Y)\prod m_i(\alpha)! \sum_{\substack{\gamma\subseteq T_\kappa \\ |\gamma|=n-k}} (-1)^{n-k}$$

(since $n-\ell(\alpha)$ = the number of rooks in $\kappa$, all of which are presently on $B$; here $|\gamma|$ is the sum of all the parts of $\gamma$)

$$= \sum_\kappa m_{\alpha(\kappa)}(X)p_{\beta(\kappa)}(Y)\prod m_i(\alpha)! \sum_{\substack{\gamma\subseteq T_\kappa \\ |\gamma|=n-k}} (-1)^{n-k} \sum_{W\subseteq E_\kappa} (-1)^{|W|}$$

(since the inner sum is 0 unless $E_\kappa=\emptyset$)

$$= \sum_\kappa m_{\alpha(\kappa)}(X)p_{\beta(\kappa)}(Y)\prod m_i(\alpha)! \sum_{\substack{S\subseteq T_\kappa\cup E_\kappa \\ |S\cap T_\kappa|=n-k}} (-1)^{|S|}$$

$$= \sum_{\substack{S \\ S \text{ has } n-k \text{ rooks on } B \\ \text{and } 0 \text{ to } k \text{ rooks on } B^c}} \sum_\kappa m_{\alpha(\kappa)}(X)p_{\beta(\kappa)}(Y)\prod m_i(\alpha)!(-1)^{|S|}$$

by reversing the order of summation. Using exactly the same argument as in [Cho, p.8], the inner sum over $\kappa$ equals

$$(-1)^{n+\ell(\alpha)} f_\alpha(X,Y) p_\beta(Y) \prod m_i(\alpha)!. \qquad \blacksquare$$

(none of the results in the rest of the paper depend on Theorem 2.1, or its special case (21), so no further details are included).

The extent to which theorems about rook polynomials and applications of reciprocity extend to the symmetric functions $C_k(B;X,Y)$ is an interesting topic for research in its own right; the focus of this article, however, is the study of the following special case of $C_k(B;X,Y)$, a new two-parameter version of the hit numbers.

**Definition** For any board $B$, define $a_k(x,y,B)$ by

$$\sum_k (x)_k r_{n-k}(y)(z-1)^{n-k} = \sum_k z^k a_{n-k}(x,y,B). \tag{20}$$

If $B$ is the triangular board (see Figure 5), the $a_k(x,1,B)$ have been introduced independently in recent work of Steingrimsson [Ste]. His approach is different from ours, involving partially ordered sets, and there is little duplication between our results.

Using known facts about symmetric functions [Cho1], one finds that if $p_k(X) \equiv -x$ and $p_k(Y) \equiv y$, $C_{n-k}(B;X,Y)$ reduces to $a_k(x,y)$. This same choice for X,Y in Theorem 2.1 then gives (for admissible $B$)

$$a_k(x,y,B) = \sum_{\substack{n-k \text{ rooks on } B \\ 0 \text{ to } k \text{ on } B^c}} (-1)^{\ell(\alpha)} (y-x)_{\ell(\alpha)} y^{\ell(\beta)}, \tag{21}$$

where $\alpha$,$\beta$ are the directed path type and cycle type of the associated rook placement. Note that

$$a_k(y,y) = \sum_{\substack{n-k \text{ rooks on } B \\ k \text{ on } B^c}} y^{\text{number of cycles}} \tag{22}$$

since $(y-y)_{\ell(\alpha)} = 0$ unless there are no directed paths, which means there are $n$ rooks on the $n \times n$ grid.

In the rest of this section we extend some of the known algebraic identities satisfied by $a_k(B)$ to $a_k(x,y,B)$.

**Theorem 2.2** (For Ferrers boards, a $q$-version of the case $j=n, x=1, y=1$ of this identity occurs in work of Garsia and Remmel [GaRe]). Let B be any board, and assume $j$ is a nonnegative integer. Then

$$\sum_{k=0}^{\infty} \binom{x+k-1}{k} a_j(-k,y,B) z^k = \frac{(-1)^j}{(1-z)^{j+x}} \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(x,y,B) z^k.$$

*Proof* : LHS times $(1-z)^{j+x} =$

$$\left(\sum_{m=0}^{\infty} z^m (-1)^m \binom{j+x}{m}\right)\left(\sum_{k=0}^{\infty} \binom{x+k-1}{k} a_j(-k,y,B) z^k\right)$$

$$= \sum_{s\geq 0} z^s \sum_{k\geq 0} \frac{(x)_k}{(1)_k} a_j(-k,y,B)(-1)^{s-k}\binom{j+x}{s-k}$$

$$= \sum_{s\geq 0} z^s \sum_{k\geq 0} \frac{(x)_k}{(1)_k}(-1)^{s-k}\binom{j+x}{s-k}\sum_{m\geq 0}(-k)_m r_{n-m}\binom{n-m}{n-j}(-1)^{j-m}$$

(by (20))

$$= \sum_{s\geq 0} z^s \sum_{m\geq 0} r_{n-m}\binom{n-m}{n-j}(-1)^{j-m+s}\sum_{k\geq 0}(-k)_m \frac{(x)_k}{(1)_k}\binom{j+x}{s}\frac{(-s)_k}{(j+x-s+1)_k}$$

(since $\binom{j+x}{s-k} = \binom{j+x}{s}(-1)^k\frac{(-s)_k}{(j+x-s+1)_k}$)

$$= \sum_{s\geq 0} z^s \binom{j+x}{s}\sum_{m\geq 0} r_{n-m}\binom{n-m}{n-j}(-1)^{j+s} m! \sum_{k\geq m}\binom{k}{m}\frac{(x)_k}{(1)_k}\frac{(-s)_k}{(j+x-s+1)_k}$$

$$= \sum_{s\geq 0} z^s \binom{j+x}{s}\sum_{m\geq 0} r_{n-m}\binom{n-m}{n-j}(-1)^{j+s} m!$$
$$\times \sum_{\substack{u\geq 0\\ k=u+m}}\binom{u+m}{m}\frac{(x)_m(x+m)_u(-s)_m(-s+m)_u}{m!(m+1)_u(j+x-s+1)_m(j+x-s+1+m)_u}$$

$$= \sum_{s\geq 0} z^s \binom{j+x}{s}\sum_{m\geq 0} r_{n-m}\binom{n-m}{n-j}(-1)^{j+s} m!$$
$$\times \frac{(x)_m(-s)_m}{m!(j+x-s+1)_m}\,{}_2F_1\left[\begin{matrix} x+m, & -s+m \\ & j+x-s+m+1\end{matrix}\right]$$

$$= \sum_{s\geq 0} z^s \binom{j+x}{s}\sum_{m\geq 0} r_{n-m}\binom{n-m}{n-j}(-1)^{j+s}$$
$$\times \frac{(x)_m(-s)_m}{(j+x-s+1)_m}\frac{(j-s+1)_{s-m}}{(j+x-s+m+1)_{s-m}}$$

(by the Vandermonde convolution (21))

$$= \sum_{s\geq 0} z^s \sum_{m\geq 0}(x)_m r_{n-m}(-1)^{j+s}\frac{(-s)_m(j-s+1)_{s-m}(j-m)_{n-j}}{(1)_s(1)_{n-j}}$$

$$= \sum_{s\geq 0} z^s \sum_{m\geq 0}(x)_m r_{n-m}(-1)^{j+s+m}\frac{(j-s+1)_{n+s-m-j}}{(1)_{s-m}(1)_{n-j}}$$

$$= \sum_{s\geq 0} z^s \sum_{m\geq 0}(x)_m r_{n-m}(-1)^{j+s-m}\frac{(n-m)!}{(s-m)!(n-j)!}\frac{(n-s)!}{(j-s)!(n-s)!}$$

$$= \sum_{s\geq 0} z^s (-1)^j \binom{n-s}{n-j} \sum_{m\geq 0} (x)_m r_{n-m} (-1)^{s-m} \binom{n-m}{n-s}$$

$= \text{RHS times } (1-z)^{j+x}$. ■

**Theorem 2.3** (The $j=n$ case of this is due to Gessel [Ges2]). For any board $B$,

$$a_j(x,y,B) = \sum_{k=0}^{j} a_k(y,y,B) \binom{n-k}{n-j} \frac{(x)_k (y-x)_{j-k}}{(y)_j} (-1)^{j-k}.$$

*Proof* : Start by setting $x=y$ in Theorem 2.2 to get

$$\sum_{k=0}^{\infty} \binom{y+k-1}{k} a_j(-k,y,B) z^k = \frac{(-1)^j}{(1-z)^{j+y}} \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(y,y,B) z^k. \qquad (23)$$

Now assume $x\in\mathbb{N}$, $x\geq j$. Then

$$a_j(-x,y,B) = \binom{y+x-1}{x}^{-1} \times \text{COEF}(z^x) \text{ in LHS of (23)}$$

$$= \binom{y+x-1}{x}^{-1} \times \text{COEF}(z^x) \text{ in} \frac{(-1)^j}{(1-z)^{j+y}} \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(y,y,B) z^k$$

$$= (-1)^j \binom{y+x-1}{x}^{-1} \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(y,y,B) \binom{j+y-1+x-k}{x-k}$$

$$= (-1)^j \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(y,y,B) \frac{x!(y+j)_{x-k}}{(x-k)!(y)_x}$$

$$= (-1)^j \sum_{k=0}^{j} \binom{n-k}{n-j} a_k(y,y,B) \frac{(-x)_k (-1)^k (y+x)_{j-k}}{(y)_j}.$$

Replacing $x$ by $-x$ proves the theorem for $x\in\mathbb{Z}$, $x\leq -j$. By definition of $a_k(x,y,B)$, it is clear that both sides are polynomials in $x$; two polynomials which have infinitely many common roots are identical. ■

In an earlier version of this paper, the author used (15) to derive Theorem 3.1, a result related to Theorem 2.7 below, but which holds for Ferrers boards only. The author is indebted to the referee for supplying him with Lemma 2.4 below (which can be thought of as a version of (15) which holds for polynomials) and for suggesting that it could be used to derive a result which holds for any admissible (not neccessarily Ferrers) board. This led to Theorem 2.7, which is analogous to (but does not contain) Theorem 3.1. Before proving it we need to derive a few lemmas; Lemma 2.5 extends well-known identities for $r_k(1,B)$ and $a_k(1,1,B)$.

**Lemma 2.4** Let $P$ be any polynomial with $\Re(c+1-w-\deg(P))>0$. Then

$$\sum_{j=0}^{\infty} \frac{(w)_j (x)_j}{j!(x+c+1)_j} P(j) = \frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)} \sum_{j=0}^{\infty} \frac{(-c)_j (x)_j}{j!(x+1-w)_j} P(-x-j).$$

*Proof* : It suffices to prove Lemma 2.4 for the basis polynomials $P(u) = (-x-c-u)_d, d = 0, 1, \ldots,$ for which both sides above can be eva luated by Gauss' theorem (eq. (11)). ■

**Lemma 2.5** Let $B$ be any board. Then

$$k! r_{n-k}(y, B) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{s=0}^{n} j(j-1)\cdots(j-s+1) r_{n-s}(y, B), \tag{24}$$

and

$$a_k(x, y, B) = \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \binom{x+j-1}{j} \sum_{s=0}^{n} j(j-1)\cdots(j-s+1) r_{n-s}(y, B). \tag{25}$$

*Proof* : The RHS of (24) equals

$$\sum_{s\geq 0} s! r_{n-s}(y) \sum_{j\geq s} \binom{k}{j} (-1)^{k-j} \binom{j}{s}$$

$$= \sum_{s\geq 0} s! r_{n-s}(y) \delta_{s,k}$$

by the Vandermonde convolution. Equation (25) can be proven similarly; the RHS of (25) equals

$$\sum_{s} r_{n-s}(y) \sum_{j\geq s} \binom{n+x}{k-j} (-1)^{k-j} \frac{(x)_j j(j-1)\cdots(j-s+1)}{(1)_j}$$

$$= \sum_{s} r_{n-s}(y) \sum_{u\geq 0, j=u+s} \binom{n+x}{k-u-s} (-1)^{k-u-s} \frac{(x)_{u+s}(u+1)_s}{(1)_{u+s}}$$

$$= \sum_{s} r_{n-s}(y) \sum_{u\geq 0} \binom{n+x}{k-s} (-1)^{u} \frac{(s-k)_u}{(n+x-k+s+1)_u} \times (-1)^{k-s-u} \frac{(x)_s (x+s)_u}{(s+1)_u} \binom{u+s}{u}$$

$$= \sum_{s} r_{n-s}(y) \binom{n+x}{k-s} (-1)^{k-s} (x)_s \sum_{u\geq 0} \frac{(-k+s)_u (x+s)_u}{(1)_u (n+x-k+s+1)_u}$$

$$= \sum_{s} (x)_s r_{n-s}(y) \binom{n+x}{k-s} (-1)^{k-s} \frac{(n-k+1)_{k-s}}{(n+x-k+s+1)_{k-s}}$$

(by the Vandermonde convolution)

$$= \sum_{s} (x)_s r_{n-s}(y) \binom{n-s}{k-s} (-1)^{k-s}. ■$$

**Lemma 2.6** For any board $B$ and $k \in \mathbb{N}$,

$$a_k(x,y,B) = \binom{n+x}{n-k}(-1)^k \sum_{j\geq 0} \frac{(k-n)_j(x)_j}{j!(x+1+k)_j}$$

$$\times \sum_{s=0}^{n}(-x-j)(-x-j-1)\cdots(-x-j-s+1)r_{n-s}(y,B).$$

*Proof* : From (25),

$$a_k(x,y,B) = \binom{n+x}{k}(-1)^k \sum_{j\geq 0} \frac{(-k)_j(x)_j}{j!(n-k+x+1)_j}P(j),$$

where $P(j)$ is the inner sum on the RHS of (25). Applying Lemma 2.4 with $w=-k$ and $c=n-k$ we get

$$a_k(x,y,B) = \binom{n+x}{k}(-1)^k \frac{\Gamma(1+x+n-k)\Gamma(k+1)}{\Gamma(1+x+k)\Gamma(n-k+1)}$$

$$\times \sum_{j\geq 0} \frac{(k-n)_j(x)_j}{j!(x+1+k)_j}P(-x-j),$$

which simplifies to Lemma 2.6. ■

**Theorem 2.7** Let $B$ be an admissible board. Then if $x-y \in \mathbb{N}$,

$$a_k(x,y,B) = a_{n-k}(x,y,D),$$

where $D$ is the board obtained by starting with $B^c$, and then affixing an $x-y$ by n rectangle with unlabelled squares to the bottom of $B^c$ (so rooks on this rectangular part do not contribute any cycles).

*Proof* : Since by (22) $a_k(y,y,B^c) = a_{n-k}(y,y,B)$, (20) implies the polynomial identity

$$\sum_k (y)_k r_{n-k}(y,B^c)z^k(1-z)^{n-k} = \sum_k (y)_k r_{n-k}(y,B)(z-1)^{n-k},$$

the case $y=1$ of which appears in [Rio]. Letting $z=z+1$ and comparing coefficients of $z^{n-s}$ on both sides we get

$$r_{n-s}(y,B) = \frac{1}{(y)_s}\sum_{m=0}^{n}(-1)^{n-m}r_{n-m}(y,B^c)(y)_m\binom{m}{m-s}.$$

Plugging this into the inner sum on the RHS of Lemma 2.6 and then reversing the sum on $m$ and $s$ we get

$$a_k(x,y,B) = \binom{n+x}{n-k}(-1)^k \sum_{j\geq 0} \frac{(k-n)_j(x)_j}{j!(x+1+k)_j} \sum_{m=0}^{n} r_{n-m}(y,B^c)(-1)^{n-m}(y)_m$$

$$\times \sum_{s=0}^{n} \frac{(-x-j)(-x-j-1)\cdots(-x-j-s+1)}{(y)_s} \binom{m}{m-s}.$$

The inner sum over $s$ can be evaluated by (12), resulting in

$$a_k(x,y,B) = \binom{n+x}{n-k}(-1)^{n-k} \sum_{j\geq 0} \frac{(k-n)_j (x)_j}{j!(x+1+k)_j}$$

$$\times \sum_{m=0}^{n} r_{n-m}(y,B^c)(x+j-y)(x+j-y-1)\cdots(x+j-y-m+1).$$

The inner sum on the RHS above can be rewritten as

$$\sum_{m=0}^{n} r_{n-m}(y,D) j(j-1)\cdots(j-m+1)$$

since by a standard argument (as in [GJW1]) both these sums count the number of ways to put $n$ non-attacking rooks on the board obtained by affixing a $j$ by $n$ rectangle to the bottom of $D$, or equivalently a $x-y+j$ by $n$ rectangle to the bottom of $B^C$, without labelling the squares of the rectangle (thus no cycles are contributed). The theorem now follows from (25) with $k$ replaced by $n-k$ and $B$ by $D$.

**Corollary 2.8** Let $B$ be any admissible board. Then for $x, y \in \mathbb{C}$

$$\sum_k (x)_k r_{n-k}(y,B)(z+1)^k(-z)^{n-k} =$$

$$\sum_s r_s(y,B^c) \sum_k (x)_k (x-y)(x-y-1)\cdots(x-y-n+k+s+1)\binom{n-s}{k} z^{n-k}.$$

*Proof* : Using (20), Theorem 2.7 can be written in polynomial form as

$$\sum_k (x)_k r_{n-k}(y,B) z^k (1-z)^{n-k} = \sum_k (x)_k r_{n-k}(y,D)(z-1)^{n-k}. \tag{26}$$

Now $r_{n-k}(y,D)$ equals

$$\sum_s r_s(y,B^c)(x-y)(x-y-1)\cdots(x-y-n+k+s+1)\binom{n-s}{k}$$

since if we wish to put $n-k$ rooks on $D$, we can put $s$ on $B^c$ in $r_s(y,B^c)$ ways, then choose $n-k-s$ of the $n-s$ columns left unattacked in $\binom{n-s}{k}$ ways, then put $n-k-s$ rooks in these selected columns in the $x-y$ by $n$ rectangular part of $D$ in $(x-y)(x-y-1)\cdots(x-y-n+k+s+1)$ ways. Using this in (26), then reversing the summation on $s$ and $k$, and finally replacing $z$ by $z+1$ yields the corollary. ■

*Remark* 1: Corollary 2.8 can be written in the following form

$$\sum_k (x)_k (z-1)^k r_{n-k}(y,B) = \sum_s r_s(y,B^c)(y-x)_{n-s}(-1)^s$$

$$\times {}_2F_1\left[\begin{matrix} s-n, & x \\ & x-y-n+s+1 \end{matrix}; z\right].$$

*Remark* 2: Comparing the coefficient of $z^n$ on both sides of Corollary 2.8 gives the identity

$$\sum_k (x)_k r_{n-k}(y,B)(-1)^{n-k} = \sum_s r_s(y,B^c)(x-y)(x-y-1)\cdots(x-y-n+s+1).$$

This is equivalent to the $k=n$ case of (21), which Chow derived from (9) and which Gessel also derived by a different method [Ges2]. Recently Chow has given a combinatorial proof of the $y=1$ case of this identity [Cho3]. Perhaps this proof could be extended to include Corollary 2.8. Another interesting issue that remains unresolved is how the identity obtained by comparing coefficients of $z^k$ for $k \neq n$ in Corollary 2.8 relates to (21), and more generally if there is a symmetric function version, along the lines of Theorem 2.1, of Corollary 2.8.

## 3. Ferrers Boards and Hypergeometric Series.

The remainder of this article will focus on Ferrers boards, for which the explicit formulas for $r_k$ and $a_k$ occurring in Lemma 2.5 can be expressed as hypergeometric series.

**Definition** Recalling that $H_i := h_1 + \dots h_i$ and $D_i := d_1 + \dots d_i$, let

$$PR(x,y,B) := \prod_{i=1}^{t} (H_i - D_i + x + y)_{d_i}.$$

*Remark* : Although $PR$ depends only on the sum $x+y$, we choose to view it as a function of both $x$ and $y$ in order to keep the connection with cycle-counting clear in what follows.

**Definition** Call a Ferrers board $B = B(h_1,d_1;h_2,d_2;...;h_t,d_t)$ *regular* if $B$ satisfies $H_i \geq D_i$ for $1 \leq i \leq t$. Also, let $e_i := H_i - D_i + y$ (we will use this notation often throughout the rest of the article!).

For $B$ a regular Ferrers board, we now convert (24) and (25) into hypergeometric notation. Note that for $j \in \mathbb{N}$,

$$(e_i + j)_{d_i} = (e_i)_{d_i} \frac{(e_i + d_i)_j}{(e_i)_j}$$

(assuming $e_i \neq 0$), hence

$$PR(j,y,B) = PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i + d_i)_j}{(e_i)_j} \qquad e_i \neq 0, 1 \leq i \leq t. \tag{27}$$

Now $H_j \geq D_j$ for $j \leq t$ implies the $i^{th}$ column of $B$ is $\geq i$ for all $i \leq n$. Thus by (3), for $B$ regular (8) can be written as

$$\sum_k x(x-1)\cdots(x-k+1) r_{n-k}(y,B) = PR(x,y,B).$$

Plugging this into (24) and (25) we get, for $k \in \mathbb{N}$,

$$k! r_{n-k}(y,B) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i+d_i)_j}{(e_i)_j}$$

$$= PR(0,y,B)(-1)^k \quad {}_{t+1}F_t \begin{bmatrix} -k, & e_1+d_1, & \dots, & e_t+d_t \\ & e_1, & \dots, & e_t \end{bmatrix}, \tag{28}$$

and

$$a_k(x,y,B) = \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \frac{(x)_j}{(1)_j} PR(0,y,B) \prod_{i=1}^{t} \frac{(e_i+d_i)_j}{(e_i)_j}$$

$$= PR(0,y,B) \binom{n+x}{k} (-1)^k$$

$$\times \quad {}_{t+2}F_{t+1} \begin{bmatrix} -k, & x, & e_1+d_1, & \dots, & e_t+d_t \\ & n+x-k+1, & e_1, & \dots, & e_t \end{bmatrix}. \tag{29}$$

*Remark* 1: Assume for the moment that $y \in \mathbb{N}$. Then clearly $PR(x,y,B) = PR(x,1,C)$, where $C = B(h_1+y-1, d_1; h_2, d_2; \dots; h_t, d_t)$ is the board obtained from $B$ by replacing $h_1$ by $h_1+y-1$. Then by (24) and (25), we see that $r_k(y,B) = r_k(1,C)$, and $a_k(x,y,B) = a_k(x,1,C)$. Now say we have an algebraic identity involving the $r_k$'s or $a_k$'s. Typically this will be a polynomial or rational function identity in the $h_i$'s and $d_i$'s. Thus it is easy to translate back and forth between identites with the $y$ parameter and those without just by changing the value of $h_1$.
*Remark* 2: The formulas above assume $B$ is regular ($H_i \geq D_i$ for $1 \leq i \leq t$). As a general rule, any formula for Ferrers boards involving the $y$ parameter in sections 3, 4, or 5 will make this same assumption. If $H_i < D_i$ for some $i$, not all of the factors on the RHS of (8) have the parameter $y$ in them. For this reason it is more convienient to work with regular boards; otherwise we can proceed by modifying the definition of $PR(x,y,B)$ appropriately. For the sake of simplicity, lets consider the case where none of the factors have a $y$ in them, and use (2) instead of (8) to get

$$a_k(x,1,B) = \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \frac{(x)_j}{(1)_j} PR(j,1,B).$$

Unfortunately, $H_i < D_i$ implies that $PR(0,1,B) = 0$, so (27) cannot be used as is. Let $u = \min j \geq 0 : PR(j,1,B) \neq 0$. Then it is easy to see that $PR(j,1,B) \neq 0$ for $j \geq u$. Thus

$$a_k(x,1,B) = \sum_{j=u}^{k} \binom{n-x}{k-j} (-1)^{k-j} PR(j,1,B)$$

$$= \sum_{\substack{s \geq 0 \\ j=u+s}} \binom{n+x}{k-s-u} (-1)^{k-s-u} \frac{(x)_{u+s}}{(1)_{u+s}} PR(u+s,1,B)$$

$$= PR(u,1,B)\binom{n+x}{k-u}(-1)^{k-u}\frac{(x)_u}{(1)_u}$$

$$\times\sum_{s\geq 0}\frac{(-k+u)_s(x+u)_s}{(n+x-k+1+u)_s(u+1)_s}\prod_{i=1}^{t}\frac{(H_i-D_{i-1}+u+1)_s}{(H_i-D_i+u+1)_s}.$$

If $H_p-D_p=\min_{(i)}\{H_i-D_i, 1\leq i\leq t\}$, then by definition of $u$, $H_p-D_p+u+1=1$, so we get

$$a_k(x,1,B)=PR(u,1,B)\binom{n+x}{k-u}(-1)^{k-u}\frac{(x)_u}{(1)_u}$$

$$\times{}_{t+2}F_{t+1}\left[\begin{matrix}-k+u, & x+u, & g_1+d_1, & \ldots, & g_p+d_p, & \ldots, & g_t+d_t\\ m, & u+1, & g_1, & \ldots, & \widehat{g_p}, & \ldots, & g_t\end{matrix}\right], \tag{30}$$

where $g_i := H_i-D_i+u+1$, and $m=n+x-k+u+1$. Similarly, $k!r_{n-k}(B)$ can be written as a ${}_{t+1}F_t$.

*Remark* 3: Since the $a_k$ are balanced, while the $k!r_{n-k}$ are not, most of our attention will be focused on the $a_k$. From results on the $a_k$ one can often deduce properties of the $r_k$ since by (28) and (29),

$$k!r_{n-k}(y)=\lim_{x\to\infty}\frac{a_k(x,y)}{\binom{n+x}{k}}.$$

Equation (8) shows that for Ferrers boards, $a_n(x,y)$ can be written as a product of linear factors in $x$ and $y$. Combining this with the $k=n$ case of (29) we get

$$\prod_{i=1}^{t}(e_i)_{d_i}\binom{n+x}{n}(-1)^n \quad {}_{t+2}F_{t+1}\left[\begin{matrix}-n, & x, & e_1+d_1, & \ldots, & e_t+d_t\\ & x+1, & e_1, & \ldots, & e_t\end{matrix}\right]$$

$$=\prod_i(e_i-x)_{d_i}. \tag{31}$$

This is equivalent to the case $w=-n$ of (14), the Karlsson-Minton summation formula. We now translate Gasper's transformation (15) into a statement about the $a_k$.

**Theorem 3.1** Let $B$ be a regular Ferrers board. Then

$$a_k(x,y,B)=a_{n-k}(x,1+x-y+n-H_t-p,\widehat{B}_p),$$

where $\widehat{B}_p$ is obtained by first rotating the $n\times H_t$ grid containing $B$ 180 degrees, keeping the squares in this grid which were not in $B$, affixing a $p\times n$ rectangle to the bottom, and finally relabelling so that the square that was $(i,j)$ is now $(n+1-i, H_t+p+1-j)$. The parameter $p$ can be any positive integer, so long as $\widehat{B}_p$ is regular. See Figure 4.

*Proof* : Let $s_i := 1+x-e_i-d_i$. By (29),

$$a_k(x,y,B)=\prod_{i=1}^{t}(e_i)_{d_i}\binom{n+x}{k}(-1)^k$$

$$\times{}_{t+2}F_{t+1}\left[\begin{matrix}-k, & x, & e_1+d_1, & \ldots, & e_t+d_t\\ & n+x-k+1, & e_1, & \ldots, & e_t\end{matrix}\right]$$

FIGURE 4. The Ferrers board $\widehat{B}_p = B(p, d_t; \ldots ; h_2, d_1)$.

$$= \prod_{i=1}^{t} (e_i)_{d_i} \binom{n+x}{k} (-1)^k \frac{\Gamma(n-x-k+1)\Gamma(k+1)}{\Gamma(n-k+1)\Gamma(k+1-x)} \prod_{i=1}^{t} \frac{(e_i - x)_{d_i}}{(e_i)_{d_i}}$$
$$\times {}_{t+2}F_{t+1} \left[ \begin{matrix} -(n-k), & x, & 1+x-e_1, & \ldots, & 1+x-e_t \\ & x+k+1, & s_1, & \ldots, & s_t \end{matrix} \right]$$

(by (15))

$$= \binom{n+x}{k} (-1)^{n-k} \frac{\Gamma(n+x-k+1)k!}{(n-k)!\Gamma(k+1+x)} \prod_{i=1}^{t} (s_i)_{d_i}$$
$$\times \quad {}_{t+2}F_{t+1} \left[ \begin{matrix} -(n-k), & x, & s_1+d_1, & \ldots, & s_t+d_t \\ & x+k+1, & s_1, & \ldots, & s_t \end{matrix} \right]. \tag{32}$$

Let $\tilde{y} = 1 + x - y + n - H_t - p$. Then

$$1 + x - e_t = 1 + x - (H_t - n + y) = p + \tilde{y} = H_1(\widehat{B}_p) - D_0(\widehat{B}_p) + \tilde{y}$$

$$1 + x - e_{t-1} = 1 + x - (H_{t-1} - D_{t-1} + y) = n - H_t + h_t - d_t + 1 + x - y$$
$$= p + h_t - d_t + \tilde{y} = H_2(\widehat{B}_p) - D_1(\widehat{B}_p) + \tilde{y}$$

$$\vdots$$

$$1+x-e_1 = 1+x-(H_1-D_1+y) = p+(h_t+\ldots+h_2)-(d_t+\ldots+d_2)+n-H_t+1+x-y-p$$
$$= H_t(\widehat{B}_p) - D_{t-1}(\widehat{B}_p) + \tilde{y}.$$

FIGURE 5. The triangular board of side $n$.

Thus

$$\text{RHS of (32)} = \binom{n+x}{n-k}(-1)^{n-k}\prod_{i=1}^{t}(\tilde{e}_i)_{\tilde{d}_i}$$

$$\times {}_{t+2}F_{t+1}\left[\begin{matrix} -(n-k), & x, & \tilde{e}_1+\tilde{d}_1, & \dots, & \tilde{e}_t+\tilde{d}_t \\ & x+k+1, & \tilde{e}_1, & \dots, & \tilde{e}_t \end{matrix}\right]$$

(where $\tilde{e}_i := H_i(\widehat{B}_p) - D_i(\widehat{B}_p) + \tilde{y}$, and $\tilde{d}_i := d_{t-i+1}(B) = d_i(\widehat{B}_p)$)

$$= a_{n-k}(x, 1+x-y+n-H_t-p, \widehat{B}_p) \qquad \text{by (29).} \qquad \blacksquare$$

**Example 3.2** Let $B$ be the triangular board of size $n$, so $B$ is regular and $H_t = n$. See Figure 5. From [EHR],

$$r_{n+1-k}(y, B) = \sum_{\substack{\lambda \\ k \text{ blocks}}} y^{num(\lambda)} := S_2(n+1, k, y)$$

say, where the sum is over all set partitions $\lambda$ of $n+1$ elements into $k$ blocks, and $num(\lambda) :=$ the number of values of $i, 1 \le i \le n$, such that the $i^{th}$ and $(i+1)^{st}$ elements are in the same block. For example $S_2(3, 2, y) = 2y + 1$ since there are 3 set partitions of $\{a, b, c\}$ into 2 blocks

$$\{a,b\}\{c\} \to y, \quad \{a,c\}\{b\} \to 1, \quad \{a\}\{b,c\} \to y.$$

Clearly $\widehat{B}_1 = B$ so after some simplification (20) and Theorem 3.1 imply

$$\sum_{k=1}^{n}(x)_{k-1}S_2(n,k,y)(z-1)^{n-k} = \sum_{k=1}^{n}(x)_{k-1}S_2(n,k,x-y)z^{k-1}(1-z)^{n-k}.$$

A version of Theorem 3.1, with $y$ set equal to 1, also holds for non-regular boards;

**Corollary 3.3** If $B$ is any Ferrers board

$$a_k(x,1,B) = a_{n-k}(x, x+n-H_t-p, \widehat{B}_p),$$

with $\widehat{B}_p$ as in Theorem 3.1.

*Proof* : Start with any regular board $C$ having the same $d_i$ as $B$. By Theorem 3.1 we have $a_k(x,y,C) = a_{n-k}(x, x+n-H_t-p, \widehat{C}_p)$. Using (25), view both sides above as polynomials in the $h_i$. If we increase a given $h_i$ by a positive integer, the equation still holds since $\widehat{C}_p$ will be regular for the same value of $p$. Thus we have two polynomials in the $h_i$, equal for all sufficiently large choices of the $h_i$, and hence equal for all $h_i$. Now let $h_i = h_i(B)$. ■

One of the well-known identities for Ferrers boards is

$$\sum_{k=0}^{\infty} PR(k,1,B) z^k = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^{n} z^k a_k(1,1,B).$$

Translating the $x$,$y$ version of this (the case $j=n$ of Theorem 2.2, together with (31)) into hypergeometric series notation led to the next result. This gives a new expression for series of Karlsson-Minton type with argument $z$, and shows they are very close to being polynomials, albeit with complicated coefficients.

**Theorem 3.4** Let $x, b_i, z \in \mathbb{C}$, $d_i \in \mathbb{N}$, and $D_t = n$. Also fix the branch of $\log z$ which is analytic for $z \in \mathbb{C}\backslash(-\infty, 0]$, with $-\pi < arg(\mathrm{z}) < \pi$ (the principal branch). Then for $z \in \mathbb{C}\backslash[1,\infty)$

$$\begin{aligned}
{}_{t+1}F_t &\left[\begin{matrix} x, & b_1+d_1, & \ldots, & b_t+d_t \\ & b_1, & \ldots, & b_t \end{matrix}; z\right] = \\
&\frac{1}{(1-z)^{n+x}} \sum_{k=0}^{n} \binom{n+x}{k} (-1)^k \\
&\times \quad {}_{t+2}F_{t+1}\left[\begin{matrix} -k, & x, & b_1+d_1, & \ldots, & b_t+d_t \\ & n+x-k+1, & b_1, & \ldots, & b_t \end{matrix}\right] z^k.
\end{aligned}$$

*Proof* : Start by assuming $|z|<1$. Expanding $(1-z)^{n+x}$ times the LHS above in powers of $z$, using absolute convergence and collecting terms,

$$(1-z)^{n+x} \sum_{j=0}^{\infty} \frac{(x)_j (b_1+d_1)_j \cdots (b_t+d_t)_j}{(1)_j (b_1)_j \cdots (b_t)_j} z^j$$

$$= \sum_k z^k \sum_{j=0}^{k} \binom{n+x}{k-j} (-1)^{k-j} \frac{(x)_j (b_1+d_1)_j \cdots (b_t+d_t)_j}{(1)_j (b_1)_j \cdots (b_t)_j}$$

$$= \sum_k z^k (-1)^k \binom{n+x}{k} \sum_{j=0}^{k} \frac{(-k)_j (x)_j (b_1+d_1)_j \cdots (b_t+d_t)_j}{(1)_j (n+x-k+1)_j (b_1)_j \cdots (b_t)_j}.$$

The inner sum above is zero for $k>n$ from the $w=-k, c=n-k$ case of (15) (since the RHS of (15) has $\Gamma(c+1)$ in the denominator). Since the RHS is just a polynomial in $z$ divided by $(1-z)^{n+x}$, the analytic continuation of the equation is immediate. ■

Set

$$Q := \frac{PR(0,y,B)^{-1}}{(1-z)^{n+x}} \sum_{k=0}^{n} a_k(x,y,B) z^k.$$

The fact that $Q$ can be expressed via Theorem 3.4 and (29) as a hypergeometric series has some other consequences. For example, it is well known that $Q$ satisfies a differential equation, namely [Bai,p.8]

$$\{D(D+e_1-1)\cdots(D+e_t-1) - z(D+x)(D+e_1+d_1)\cdots(D+e_t+d_t)\}Q = 0,$$

where $D$ is the operator $z\frac{d}{dz}$. Also, Q can be expressed as an integral in various ways [Bai], [Erd], [Kar], [MacR]. For example, a result of Erdélyi can be phrased as

$$F(x,z,B) = (1-z)^{n+x} d_t!(-1)^{d_t} \frac{i}{2\pi} \int\limits_C \frac{F(x,z\rho,B')}{(1-\rho)^{d_t+1}(1-z\rho)^{n-d_t+x}} d\rho,$$

with $C$ any closed contour circling the point $\rho = 1$ counterclockwise, $B'$ the truncated board $B' = B(h_1,d_1;\dots;h_{t-1},d_{t-1})$, and $F(x,z,B) := \sum_{k=0}^{n} a_k(x,y,B)z^k$.

Figure 6. The board for a well-poised series.

*Well-poised boards*

A ${}_{t+1}F_t\left[\begin{matrix} x, & c_1, & \dots, & c_t \\ & b_1, & \dots, & b_t \end{matrix}; z\right]$ is said to be well-poised if $x+1 = b_1 + c_1 = \dots = b_t + c_t$. Together with balanced series they form the most important class of hypergeometric series. When expressed in terms of the $a_k$, the well-poised case of (15) has a particularly simple form;

$$a_k(x, 1, B) = a_{n-k}(x, 1, B), \tag{33}$$

with $B$ the board of Figure 6.

There are analogoues of (16) for well-poised series, due to Sears and Slater, as well as for well-poised bilateral series (see Chapters 4 and 5 of [GaRa]). We will not do a systematic exploration of the Karlsson-Minton cases of these at this time, leaving this as a topic for future research. Instead we content ourselves with listing a result on well-poised series which is an easy corollary of Gaspers transformation.

**Theorem 3.5** Let $d_i \in \mathbb{N}$, $n$ odd, set $w_i := (1 + x - d_i)/2$. Then for $x \in \mathbb{C}$,

$$ {}_{t+2}F_{t+1}\left[\begin{matrix} w, & x, & w_1 + d_1, & \dots, & w_t + d_t \\ & -w + x + 1, & w_1, & \dots, & w_t \end{matrix}; 1\right] = 0, \tag{34}$$

where $\Re(-2w + 1 - n) > 0$, and

$$ {}_{t+1}F_t\left[\begin{matrix} x, & w_1 + d_1, & \dots, & w_t + d_t \\ & w_1, & \dots, & w_t \end{matrix}; -1\right] = 0, \tag{35}$$

where $\Re(-x - n) > 0$.

*Proof* : Letting $c = -w$ and $b_i = w_i$ in (15) the series in (15) become well-poised and we get LHS of (34) = $(-1)^n$ times LHS of (34). This implies (34) since $n$ is odd. Equation (35) will follow from the $q$-version of (15) discussed in section 5 (see the remark following (57)).

**4. Generating Functions and Recurrence Relations.**

Two ${}_pF_t$'s are called *contiguous* if they differ by exactly one in exactly one parameter (except that they must have the same argument $z$). Gauss derived the contiguous relations for the ${}_2F_1$, and Rainville [Rai1] did the general ${}_pF_t$. We use an abbreviated notation indicated as follows

$$F = {}_{t+1}F_t \left[ \begin{matrix} x, & c_1, & \dots, & c_t \\ & b_1, & \dots, & b_t \end{matrix} ; z \right], F(x+) = {}_{t+1}F_t \left[ \begin{matrix} x+1, & c_1, & \dots, & c_t \\ & b_1, & \dots, & b_t \end{matrix} ; z \right],$$

$$F(b_1-) = {}_{t+1}F_t \left[ \begin{matrix} x, & c_1, & \dots, & c_t \\ & b_1 - 1, & \dots, & b_t \end{matrix} ; z \right].$$

In the case $p = t+1$, the simplest of the Rainville recurrence relations are

$$(x - c_k)F = xF(x+) - c_k F(c_k+), \qquad k = 1, ..., t$$

and

$$(x - b_k + 1)F = xF(x+) - (b_k - 1)F(b_k-), \qquad k = 1, ..., t \tag{36}$$

as well as relations with $x$ and $c_i$ interchanged. There are also a set of $p+1$ linearly independent relations of a more complicated nature, each involving $t+2$ contiguous functions.

Theorem 3.4 can be restated in the form

$$PR(0, y, B)(1-z)^{n+x} \quad {}_{t+1}F_t \left[ \begin{matrix} x, & e_1 + d_1, & \dots, & e_t + d_t \\ & e_1, & \dots, & e_t \end{matrix} ; z \right] =$$

$$\sum_{k=0}^{n} z^k a_k(x, y, B). \tag{37}$$

Hence any contiguous relation satisfied by the ${}_{t+1}F_t$ on the LHS above can be translated into a recurrence involving the $a_k$. For future reference we list some of the relations obtained by this procedure below; the proofs are routine, so they are omitted. The notation $B+h_i+d_j$ refers to the board obtained from $B$ by increasing $h_i$ and $d_j$ by one each, and leaving the other parameters fixed. Similar remarks apply to $B - h_i - d_j$. Also, let $f_i := e_i + d_i$.

Under the assumption that all boards underappearing in the following formulas are regular, the first of Rainville's relations yields (there are two cases to consider, since $x$ is a different type of parameter then the $f_i$)

$$a_k(x, y, B + h_i + d_i) = xa_k(x+1, y, B) + (f_i - x)a_k(x, y, B) - (f_i - x)a_{k-1}(x, y, B)$$

and

$$(f_i - f_s)(a_k(x, y, B) - a_{k-1}(x, y, B) = a_k(x, y, B + h_i + d_i) - a_k(x, B + h_s + d_s),$$

where $1 \le i, s \le t, 0 \le k \le n+1$. The second simple relation gives

$$(x - e_i + 1)(a_k(x, y, B) - a_{k-1}(x, y, B)) = xa_k(x+1, y, B) - a_k(x, y, B + d_i + h_{i+1})$$

for $1 \le i \le t, h_{t+1} = 0$, and

$$(f_j - e_i + 1)(a_k(x, y, B) - a_{k-1}(x, y, B)) = a_k(x, y, B + h_j + d_j) - a_k(x, B + d_i + h_{i+1})$$

where $1 \le i, j \le t$. The first type of more complicated contiguous relation ([Rai2, Eq. (28), p.84]) translates into

$$xa_k(x, y, B) + na_{k-1}(x, y, B) = xa_k(x+1, y, B) +$$
$$\sum_{j=1}^{t} (e_j - x) \frac{\prod_{s=1}^{t} (f_s - e_j)}{\prod_{\substack{s=1 \\ s \ne j}}^{t} (e_s - e_j)} (a_{k-1}(x, y, B - d_j - h_{j+1}) - a_{k-2}(x, y, B - d_j - h_{j+1}),$$

and

$$f_m a_k(x, y, B) + (n + x - f_m) a_{k-1}(x, y, B) = a_k(x, y, B + d_m + h_m) +$$
$$\sum_{j=1}^{t} (e_j - x) \frac{\prod_{s=1}^{t} (f_s - e_j)}{\prod_{\substack{s=1 \\ s \ne j}}^{t} (e_s - e_j)} (a_{k-1}(x, y, B - d_j - h_{j+1}) - a_{k-2}(x, y, B - d_j - h_{j+1}),$$

where $1 \le m \le t$. The second type of more complicated relation ([Rai2, Eq. (30), p.85]) gives

$$a_k(x, y, B) = a_k(x-1, y, B) +$$
$$\sum_{j=1}^{t} \frac{\prod_{s=1}^{t} (f_s - e_j)}{\prod_{\substack{s=1 \\ s \ne j}}^{t} (e_s - e_j)} (a_{k-1}(x, y, B - d_j - h_{j+1}))$$

for $0 \le k \le n$, and

$$a_k(x, y, B) = (f_i - 1) a_k(x, y, B - d_i - h_i) +$$
$$\sum_{j=1}^{t} (x - e_j) \frac{\prod_{\substack{s=1 \\ s \ne i}}^{t} (f_s - e_j)}{\prod_{\substack{s=1 \\ s \ne j}}^{t} (e_s - e_j)} a_{k-1}(x, y, B - d_j - h_{j+1}),$$

where $1 \le i \le t$, and $0 \le k \le n$. Note that the relations above can be viewed as statements involving balanced series via (29).

In the remainder of this section we investigate recurrence relations which can be derived by combinatorial methods. Some identities best described as "iterated" contiguous relations for balanced series are obtained. We also show that special cases of Saalschütz summation and Whipple's ${}_4F_3$ transformation have simple combinatorial interpretations involving permutations of multisets.

By exploiting a connection between compositions of vectors and rook placements [Hag2, Thm. 22], the following generating function for rook polynomials of Ferrers boards is obtained

$$(\sum_{i=1}^{t} x_i + y_i + \sum_{1 \le i \le j \le t} x_i y_j)^k = \sum_{\mathbf{h}, \mathbf{d} \in \mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} r_{H_t + D_t - k}(B(h_1, d_1; \ldots ; h_t, d_t)) k!.$$

Using this we derive a generating function for $a_k(x,1,B)$;

$$(1-\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i + (1-z)\sum_{1\le i\le j\le t} x_i y_j)^{-x}$$

$$= \sum_{k=0}^{\infty} \binom{-x}{k} (-1)^k (\sum_{i=1}^{t} x_i + y_i + (z-1)\sum_{1\le i\le j\le t} x_i y_j)^k$$

$$= \sum_{k=0}^{\infty} \frac{(x)_k}{k!(z-1)^k} (\sum_{i=1}^{t} (z-1)x_i + (z-1)y_i + \sum_{1\le i\le j\le t} (z-1)x_i(z-1)y_j)^k$$

$$= \sum_{k=0}^{\infty} \frac{(x)_k}{k!} \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} (z-1)^{H_t+D_t-k} r_{H_t+D_t-k}(B(h_1,d_1;\dots))k!$$

$$= \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} \sum_{k\ge 0} (x)_k (z-1)^{H_t+D_t-k} r_{H_t+D_t-k}(B(h_1,d_1;\dots))$$

$$= \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} (x)_{H_t} \sum_{s\ge 0, k=H_t+s} (x+H_t)_s (z-1)^{n-s} r_{n-s}(B(h_1,d_1;\dots)),$$

since $r_j = 0$ if $j > n$. As usual, $n = D_t$. Thus by (20),

$$(1-\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i + (1-z)\sum_{1\le i\le j\le t} x_i y_j)^{-x}$$

$$= \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} (x)_{H_t} \sum_{k=0}^{n} a_{n-k}(x+H_t,1,B) z^k. \tag{38}$$

By differentiating (38) we can derive recurrence relations for the $a_k$. For example, if we differentiate with respect to $z$ we get

$$-x(1-\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} y_i + (1-z)\sum_{1\le i\le j\le t} x_i y_j)^{-x-1}(-\sum_{i\le j} x_i y_j)$$

$$= (-x \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} (x+1)_{H_t} \sum_{k=0}^{n} a_{n-k}(x+1+H_t,1,B) z^k)(-\sum_{i\le j} x_i y_j)$$

$$= \sum_{\mathbf{h},\mathbf{d}\in\mathbb{N}^t} \prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} (x)_{H_t} \sum_{k=0}^{n} a_{n-k}(x+1+H_t,1,B) z^{k-1} k.$$

Comparing coefficients of $\prod_{i=1}^{t} \frac{x_i^{h_i} y_i^{d_i}}{h_i! d_i!} z^k$ in the last two lines above gives

$$\sum_{1\le i\le j\le t} x(x+1)_{H_t-1} a_{n-1-k}(x+H_t,1,B-h_i-d_j) h_i d_j$$
$$= (k+1)(x)_{H_t} a_{n-k-1}(x+H_t,1,B).$$

Replacing $k$ by $n-1-k$ this becomes

$$(n-k)a_k(x,1,B) = \sum_{1\le i\le j\le t} h_i d_j a_k(x,1,B-h_i-d_j). \tag{39}$$

The $t=1$ case of (39) yields another proof of (13); assuming $B$ is regular, the $y=1$ case of (29) implies, with $b := H_1 - D_1 + 1$, and $n := d_1$,

$$(n-k)\binom{n+x}{k}(-1)^k (b)_n \quad {}_3F_2\left[\begin{matrix}-k, & x, & b+n \\ & n+x-k+1, & b\end{matrix}\right]$$

$$= n(b)_n\binom{n+x-1}{k}(-1)^k {}_3F_2\left[\begin{matrix}-k, & x, & b+n-1 \\ & n-1+x-k+1, & b\end{matrix}\right]. \tag{40}$$

Although in the proof of (40) we assumed $n$ and $b$ are positive integers, since both sides are rational functions of $n$ and $b$, (40) holds for $n$,$b \in \mathbb{C}$. After cancelling common factors, iterating $m$ times and taking the limit as $m$ approaches infinity we get, with $f := b+n$,

$${}_3F_2\left[\begin{matrix}-k, & x, & f \\ & b, & f+x-k-b+1\end{matrix}\right]$$

$$= \lim_{m\to\infty} \frac{(-x+k+b-f)_m (b-f)_m}{(k+b-f)_m(-x+b-f)_m} {}_3F_2\left[\begin{matrix}-k, & x, & f-m \\ & b, & f-m-b-k+x+1\end{matrix}\right]$$

$$= \frac{\Gamma(k+b-f)\Gamma(-x+b-f)}{\Gamma(-x+k+b-f)\Gamma(b-f)} \quad {}_2F_1\left[\begin{matrix}-k, & x \\ & b\end{matrix}\right]$$

$$= \frac{(b-f)_k(b-x)_k}{(b-x-f)_k(b)_k}$$

(by the Vandermonde convolution) which is (13).

One can derive other relations by differentiating (38) with respect to $x$, $x_p$, or $y_p$ for $1 \le p \le t$. Differentiating with respect to $x$ produces a logarithm on the LHS, which doesn't yield an easily describable relation. Below we list the relations you get by differentiating with respect to $x_p$ and $y_p$, $1 \le p \le t$, omitting the details. Differentiating with respect to $x_p$;

$$a_k(x,1,B+h_p) = a_k(x,1,B) + \sum_{j=p}^{t} d_j(a_k(x,1,B-d_j) - a_{k-1}(x,1,B-d_j)).$$

Differentiating with respect to $y_p$;

$$a_{k+1}(x,1,B+d_p) = xa_k(x+1,1,B) + \sum_{j=1}^{p} h_j(a_{k+1}(x,1,B-h_j) - a_k(x,1,B-h_j)).$$

Using (37), (38) can be expressed as

$$(1 - \sum_{i=1}^{t} x_i - z\sum_{i=1}^{t} y_i + (z-1)\sum_{1\le i\le j\le t} x_i y_j)^{-x} =$$

$$\sum_{\mathbf{h},d\in\mathbb{N}^t}\prod_{i=1}^{t}\frac{x_i^{h_i}y_i^{d_i}}{h_i!d_i!}(x)_{H_t}(1-z)^{n+x+H_t}$$
$$\times{}_{t+1}F_t\left[\begin{matrix}x+H_t, & c_1+d_1, & \dots, & c_t+d_t\\ & c_1, & \dots, & c_t\end{matrix};z\right]PR(0,1,B), \tag{41}$$

where $c_i := H_i - D_i + 1$. Actually the RHS of (41) is not completely correct as written, since if $c_i < 0$ for some $i$, the ${}_{t+1}F_t$ has to be shifted as in (30). We can obtain recurrence relations for series with argument $z$ as before by differentiating (41). For example, if we differentiate with respect to $x_p$, after simplification we get

$$\prod_{j=p}^{t}\frac{f_j}{b_j}\qquad {}_{t+1}F_t\left[\begin{matrix}w, & f_1, & \dots, & f_{p-1}, & f_p+1, & \dots, & f_t+1\\ & b_1, & \dots, & b_{p-1}, & b_p+1, & \dots, & b_t+1\end{matrix};z\right]$$
$$={}_{t+1}F_t\left[\begin{matrix}w, & f_1, & \dots, & f_t\\ & b_1, & \dots, & b_t\end{matrix};z\right]$$
$$+\sum_{j=p}^{t}\frac{d_j}{b_j}\prod_{i=j+1}^{t}\frac{f_i}{b_i}{}_{t+1}F_t\left[\begin{matrix}w, & f_1, & \dots, & f_{j-1}, & f_j+1, & \dots, & f_t+1\\ & b_1, & \dots, & b_{j-1}, & b_j+1, & \dots, & b_t+1\end{matrix};z\right], \tag{42}$$

where $b_0 = 1, b_i \in \mathbb{C}, d_i \in \mathbb{N}, n = D_t$, $|z| < 1$, or $|z| = 1$ and $\Re(-w+\sum_i b_i - f_i) > 0$.

Michael Schlosser [Sch] has found a complicated formula, involving many different $q$-parameters, which includes the $z = 1$ case of (42). He has also noted that (42) holds for $d_i \in \mathbb{C}$, and provided the following simple proof. Comparing coefficients of $z^k$ on both sides, and pulling out common terms, we have only to check that

$$\prod_{j=p}^{t}\frac{f_j+k}{b_j+k} = 1+\sum_{j=p}^{t}\frac{f_j-b_j}{b_j+k}\prod_{i=j+1}^{t}\frac{f_i+k}{b_i+k}.$$
$$=1+\sum_{j=p}^{t}\{\prod_{i=j}^{t}\frac{f_i+k}{b_i+k}-\prod_{i=j+1}^{t}\frac{f_i+k}{b_i+k}\}$$

which telescopes to the LHS. ■

A different type of recurrence for the $a_k$ can be derived by starting with the $j = n$ case of Theorem 2.2. Letting $C_p$ be the board $B - h_p - d_p$,

$$\frac{1}{(1-z)^{n+x}}\sum_{k=0}^{n}a_k(x,y,B(h_1,d_1;\dots))z^k=\sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,B)z^k$$
$$=\sum_{k=0}^{\infty}\frac{(x)_k}{k!}(e_p+k)_{d_p-1}(e_p+d_p+k-1)\prod_{i\geq 1,i\neq p}(e_i+k)_{d_i}z^k$$
$$=\sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,C_p)kz^k+(e_p+d_p-1)\sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,C_p)$$
$$=z\frac{d}{dz}\frac{1}{(1-z)^{n-1+x}}\sum_{k=0}^{n-1}a_k(x,y,C_p)z^k+\frac{(e_p+d_p-1)}{(1-z)^{n-1+x}}\sum_{k=0}^{n-1}a_k(x,y,C_p)z^k$$

$$= z\{\frac{-(-x+1-n)}{(1-z)^{n+x}}\sum_{k=0}^{n-1} a_k(x,y,C_p)z^k + \frac{1}{(1-z)^{n-1+x}}$$
$$\times \sum_{k=0}^{n-1} a_k(x,y,C_p)kz^{k-1}\} + \frac{(e_p+d_p-1)}{(1-z)^{n-1+x}}\sum_{k=0}^{n-1} a_k(x,y,C_p)z^k.$$

Multiplying both sides by $(1-z)^{n+x}$ and comparing coefficients of $z^k$ we get

$$a_k(x,y,B) = -(-x+1-n)a_{k-1}(x,y,C_p) + ka_k(x,y,C_p)$$
$$-(k-1)a_{k-1}(x,y,C_p) + (e_p+d_p-1)(a_k(x,y,C_p) - a_{k-1}(x,y,C_p))$$
$$= a_k(x,y,C_p)(k+e_p+d_p-1) + a_{k-1}(x,y,C_p)(n+x-k-e_p-d_p+1)$$
$$= a_k(x,y,C_p)(k+e_p+d_p-1) + a_{k-1}(x,y,C_p)(n-e_p-d_p+x-k+1). \quad (43)$$

**Theorem 4.1** If $B_j = B(h_1,d_1;\ldots;h_{p-1},d_{p-1};h_p-j,d_p-j;h_{p+1},d_{p+1};\ldots;h_t,d_t)$ is the board obtained from a regular Ferrers board $B$ by decreasing $h_p$ and $d_p$ by $j$ (here we assume $j \geq h_p, d_p$), then

$$a_k(x,y,B) = j!\sum_{s=k-j}^{k} a_s(x,y,B_j)\binom{e_p+d_p+s-1}{s-k+j}\binom{n-e_p-d_p-s+x}{k-s}.$$

*Proof* : By induction on $j$. The case $j=0$ is trivial, and the case $j=1$ is (43). By the induction hypothesis, abbreviating $e_p+d_p$ by $T_p$,

$$a_k(x,y,B) = (j-1)!\sum_{s=k-j+1}^{k} a_s(x,y,B_{j-1})\binom{T_p+s-1}{s-k+j-1}\binom{n-T_p-s+x}{k-s}.$$

Next apply (43) to $a_s(x,y,B_{j-1})$ to get

$$a_k(x,y,B) = (j-1)!\sum_{s=k-j+1}^{k}\{a_s(x,y,B_j)(s+e_p+d_p-(j-1)-1)$$
$$+a_{s-1}(x,y,B_j)(n-(j-1)-(e_p+d_p-(j-1))+x-s+1)\}$$
$$\times\binom{e_p+d_p+s-1}{s-k+j-1}\binom{n-e_p-d_p-s+x}{k-s}$$
$$= (j-1)!\sum_{s=k-j}^{k} a_s(x,y,B_j)\binom{e_p+d_p+s-1}{s-k+j}\binom{n-e_p-d_p-s+x}{k-s}$$
$$\times\{\frac{(j-k+s)(e_p+d_p-j+s)}{e_p+d_p+s-(j-k+s)} + \frac{(e_p+d_p+s)(k-s)}{e_p+d_p+s-(j-k+s)}\}.$$

The numerator of the expression inside the brackets is $j$ times the denominator, which completes the proof. ■

A modification of the method leads to a slightly different recurrence;

$$\frac{1}{(1-z)^{n+x}}\sum_{k=0}^{n} a_k(x,y,B(h_1,d_1;\dots))z^k = \sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,B)z^k$$

$$= \sum_{k=0}^{\infty}\frac{(x)_k}{k!}(e_p+k+1)_{d_p-1}(e_p+k)z^k \prod_{i\geq 1, i\neq p}(e_i+k)_{d_i}$$

$$= e_p\sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,B-d_p-h_{p+1})z^k + z\frac{d}{dz}\sum_{k=0}^{\infty}\frac{(x)_k}{k!}PR(k,y,B-d_p-h_{p+1})$$

(if $p=t, h_{p+1}=0$). Proceeding as before we end up with

$$a_k(x,y,B) = a_k(x,y,B-d_p-h_{p+1})(k+H_p-D_p+y)$$

$$+a_{k-1}(x,y,B-d_p-h_{p+1})(n-H_p+D_p-y+x-k). \tag{44}$$

**Theorem 4.2** Let $B$ be a regular Ferrers board. Let $B_{(j)} = B(h_1,d_1;\dots;h_p,d_p-j;h_{p+1}-j,d_{p+1};\dots;h_t,d_t)$ be the board obtained from $B$ by decreasing $d_p$ and $h_{p+1}$ by $j$ (here we assume $j \leq d_p, h_{p+1}$). Also, if $p=t$, $j\leq d_t$, and $B_{(j)}$ is the board obtained from $B$ by decreasing $d_t$ by $j$. Then

$$a_k(x,y,B) = j!\sum_{s=k-j}^{k} a_s(x,y,B_{(j)})\binom{e_p+j+s-1}{s-k+j}\binom{n-e_p-s-j+x}{k-s}.$$

*Proof* : By induction on $j$. The case $j=0$ is trivial, and the case $j=1$ is (44). By the induction hypothesis,

$$a_k(x,y,B) =$$
$$(j-1)!\sum_{s=k-j+1}^{k} a_s(x,y,B_{(j-1)})\binom{e_p+j-1+s-1}{s-k+j-1}\binom{n-e_p-s+x-j+1}{k-s}.$$

Next apply (44) to $a_s(x,y,B_{(j-1)})$ to get

$$a_k(x,y,B) = (j-1)!\sum_{s=k-j+1}^{k}\{a_s(x,y,B_{(j)})(s+e_p-(-(j-1)))$$

$$+a_{s-1}(x,y,B_{(j)})(n-(j-1)-(j-1)-e_p+x-s)\}$$
$$\times\binom{e_p+j+s-2}{s-k+j-1}\binom{n-e_p-s+x-j+1}{k-s}$$

$$= (j-1)!\sum_{s=k-j}^{k} a_s(x,y,B_{(j)})\binom{e_p+j+s-1}{s-k+j}\binom{n-e_p-s-j+x}{k-s}$$

$$\times\{\frac{(j-k+s)(n-e_p+x-s-j+1)}{n-e_p+x-s-j-(k-s)+1}+\frac{(k-s)(n-2j-e_p+x+1-s)}{n-e_p+x-s-j-(k-s)+1}\}.$$

As before, the expression inside the brackets is $j$. ■

**Corollary 4.3** Let $B' = B(h_1, d_1; \dots ; h_{p-1}, d_{p-1}; h_p + h_{p+1} - d_p, d_{p+1}; \dots ; h_t, d_t)$ be the Ferrers board obtained from $B$ by removing the "$p^{th}$ step" (if $p = t$, $B' = B(h_1, d_1; \dots ; h_{t-1}, d_{t-1})$). Assume $d_p \le h_p + h_{p+1}$, or that $p = t$. Then

$$a_k(x, y, B) = d_p! \sum_{s=k-d_p}^{k} a_s(x, y, B') \binom{e_p + d_p + s - 1}{s - k + d_p} \binom{n - e_p - d_p - s + x}{k - s},$$

where $a_s(x, y, \emptyset) = \delta_{s,0}$.
*Proof* : Set $j = d_p$ in Theorem 4.1, or in Theorem 4.2. ■
*Remark* : Corollary 4.3, phrased in terms of hypergeometric series, is due to MacRobert [MacR, Eq. (30), p.365]. It shows how to express a terminating, balanced ${}_{t+2}F_{t+1}$ in terms of terminating, balanced ${}_{t+1}F_t$'s. If we let $t = 1$, then $B' = \emptyset$, and there is only one term on the RHS. After simplification, this reduces to the Pfaff-Saalschütz summation formula mentioned earlier. Letting $t = 2$, and using the fact that $a_s(x, y, B')$ can be summed, the RHS turns out to be a terminating, balanced ${}_4F_3$ (as does the LHS). This theorem is known as $Whipple's\ transformation$ [Bai]. MacRobert derived this from a multisum identity of his [MacR, p.363], which can be rephrased as the following recurrence

$$ {}_{t+1}F_t \left[ \begin{matrix} x, & c_1, & \dots, & c_t \\ & b_1, & \dots, & b_t \end{matrix} ; z \right] = \sum_{m=0}^{\infty} \frac{(b_1 - x)_m (b_1 - c_1)_m (c_2)_m \cdots (c_t)_m}{m! (b_1)_m \cdots (b_t)_m} z^m $$

$$ \times \quad {}_tF_{t-1} \left[ \begin{matrix} x + c_1 - b_1, & c_2 + m, & \dots, & c_t + m \\ & b_2 + m, & \dots, & b_t + m \end{matrix} ; z \right], $$

valid for $|z| < 1$. Note that if $c_1 = b_1 + d_1, d_1 \in \mathbb{N}$, the RHS above reduces to a finite sum of ${}_tF_{t-1}$'s.

The $x = 1$, $y = 1$ case of Corollary 4.3 was previously discovered by the author [Hag1], in connection with the study of $permutations\ of\ multisets$. A permutation $\sigma$ of a multiset $M$ is a linear list $\sigma_1 \sigma_2 \cdots \sigma_{|M|}$ of the elements of $M$. Let $N_k(\mathbf{v}; r)$ be the number of permutations of the multiset in which $i$ occurs $v_i$ times, having exactly $k - 1$ $r$-descents. An $r$-descent is a value of $i$, $1 \le i \le |M| - 1$, such that $\sigma_i - \sigma_{i+1} \ge r$. For example, if $\mathbf{v} = (2, 1, 1)$, then there are 12 permutations in question;

$$a)3211 \quad b)3121 \quad c)3112 \quad d)2311 \quad e)2131 \quad f)2113$$

$$g)1321 \quad h)1312 \quad i)1231 \quad j)1213 \quad k)1132 \quad l)1123.$$

Permutations $b$, $c$, $d$, $e$, $h$, and $i$ all have one 2-descent, and the others have no 2-descents. Thus $N_2((2, 1, 1); 2) = 6$ and $N_1((2, 1, 1); 2) = 6$. Also, $N_3((2, 1, 1); 1) = 4$, $N_2((2, 1, 1); 1) = 7$, and $N_1((2, 1, 1); 1) = 1$.

In [Hag1,p. 118] it is shown that

$$N_k(\mathbf{v}; r) \prod_{i=1}^{t} v_i! = a_{n+1-k}(G_{\mathbf{v},r}), \tag{45}$$

where $G_{\mathbf{v},r}$ is the Ferrers board of Figure 7. Applying Corollary 4.3 to this board, using (45), and setting $y = 1$ gives the following identity

$$N_k(x, \mathbf{v}, r) = \sum_{s} N_s(x, \mathbf{v}', r)$$

FIGURE 7. The Ferrers board $G_{\mathbf{v},r}$

$$\times\binom{v_1+\ldots+v_{t-r}+1-s}{k-s}\binom{v_{t-r+1}+\ldots+v_t+s+x-2}{v_t-k+s}, \tag{46}$$

where $n=v_1+\ldots+v_t$, $\mathbf{v}'=(v_1,\ldots,v_{t-1})$, and by definition

$$N_k(x,\mathbf{v},r):=(-1)^{n-k+1}\sum_{s=0}^{n-k+1}N_{k+s}(\mathbf{v},r)\binom{s+k-1}{s}\binom{-x+s}{n-k+1}.$$

If $x=1$ and $r=1$, (46) reduces to

$$N_k(\mathbf{v},1)=\sum_{s=0}^{v_t}N_{k-s}(\mathbf{v}',1)\binom{v_t+k-1-s}{v_t-s}\binom{n-v_t-k+1+s}{s},$$

which can be proven rather easily by a direct combinatorial argument [DiR].

## 5. q-Versions.

In 1986 Garsia and Remmel [GaRe] introduced a $q$-version of rook theory for Ferrers boards. Throughout this section, let $q$ be a real variable satisfying $0<q<1$. They define

$$R_k(B):=\sum_{\text{placements } C \text{ of } k \text{ rooks on } B}q^{invC},$$

where $inv\ C$ is a certain statistic. To calculate it, cross out all squares on $B$ below and all squares on $B$ to the right of each rook in $C$. The number of squares on $B$ not crossed out by this procedure is $inv\ C$.
*Example* : If $C$ consists of rooks on squares $(2,1)$ and $(4,3)$ of the $n=4$ case of Figure 5, then $inv\ C=5$.

This definition led them to a $q$-version of (2)

$$\sum_{k=0}^{n}[x][x-1]\cdots[x-k+1]R_{n-k}(B)=\prod_{i=1}^{n}[x+c_i-i+1],$$

where $[x] := \frac{1-q^x}{1-q}$ (which approaches $x$ as $q \to 1$) and $c_i$ is the height of the $i^{th}$ column of $B$.
Garsia and Remmel also define a $q$-version of $a_k(B)$ as follows

$$\sum_{k=0}^{n} R_{n-k}(B)[k]!z^k \prod_{i=k+1}^{n}(1-zq^i)=\sum_{k=0}^{n}A_k(B),$$

where $[k]! := \prod_{i=1}^{k}[i]$. The polynomial $A_k(B)$ equals $a_k(B)$ when $q=1$.

A conjecture they made, that $R_k(B)$ is a unimodal polynomial in $q$ for all $k$ and all Ferrers boards, is still open. They were able to show that for admissible $B$, $A_k(B) \in \mathbb{N}[q]$, and in [Hag1] it was demonstrated that their proof extends easily to show that for such boards $A_k(B)$ is a symmetric and unimodal polynomial in $q$.

In the special case of the triangular board, the polynomials $R_k(B)$ are $q$-versions of the Stirling numbers of the second kind. Wachs and White have introduced the study of $p$,$q$-Stirling numbers [WaW], which are polynomials in two variables $p$ and $q$, and which can be defined as sums over rook placements on the triangular board. An interesting open question is whether or not there is a $p$,$q$-version of rook polynomials for general Ferrers boards with significant properties.

A cycle-counting version of $R_k$ has been introduced in [EHR]. The following fact is used in its description: given a placement of $j$ non-attacking rooks in columns 1 through $i$ of $B$, where $0 \le j \le i$, then if $c_i$ (the height of the $i^{th}$ column of $B$) is $\ge i$, there is one and only one square in column $i$ where a rook placement will complete a cycle. If $c_i < i$, there is no such square.

Given a placement $C$ of $k$ rooks on $b$, define $s_i$ as follows; if $c_i < i$, $s_i =$ the square $(i, c_i+1)$, while if $c_i \ge i$, $s_i$ is the unique square such that, considering only the rooks from $C$ in columns 1 through $i-1$, a rook on square $s_i$ completes a cycle. Set $E = E(C) =$ the number of $i$ such that there is a rook from $C$ in column $i$ on or above square $s_i$. Then if we define

$$R_k(y,B) := \sum_{\substack{C \\ k \text{ rooks}}} [y]^{\text{number of cycles of C}} q^{invC+E}$$

then [EHR]

$$\begin{aligned}&\sum_{k=0}^{n}[x][x-1]\cdots[x-k+1]R_{n-k}(y,B)\\ &=\prod_{c_i\ge i}[x+c_i-i+y]\prod_{c_i<i}[x+c_i-i+1] \qquad (47)\\ &=PR[x,y,B]\end{aligned}$$

say. We can use $R_k(y,B)$ to define a $q$-version of $a_k(x,y,B)$;

$$\sum_{k=0}^{n}[x][x+1]\cdots[x+k-1]R_{n-k}(y,B)z^k \prod_{i=k+1}^{n}(1-zq^{i+x-1}) := \sum_{k=0}^{n}A_k(x,y,B)z^k.$$

An easy calculation shows that if $q \to 1$, $A_k(x,y,B) \to a_k(x,y,B)$.

The Chu-Gasper transformation for Karlsson-Minton type series discussed in section 1 is

$$
{}_{t+2}\phi_{t+1}\left(\begin{matrix} w, & \alpha, & b_1q^{d_1}, & \ldots, & b_tq^{d_t} \\ & qc\alpha, & b_1, & \ldots, & b_t \end{matrix}; q^{1-j}/w\right) = \alpha^j \frac{(q\alpha/w;q)_\infty (qc;q)_\infty}{(qc\alpha;q)_\infty (q/w;q)_\infty}
$$

$$
\times \prod_{k=1}^{t} \frac{(b_k/\alpha;q)_{d_k}}{(b_k;q)_{d_k}} {}_{t+2}\phi_{t+1}\left(\begin{matrix} 1/c, & \alpha, & q\alpha/b_1, & \ldots, & q\alpha/b_t \\ & q\alpha/w, & q^{1-d_1}\alpha/b_1, & \ldots, & q^{1-d_t}\alpha/b_t \end{matrix}; q^{1+j-n}c\right) \tag{48}
$$

with $n = D_t$, valid for $j \in \mathbb{Z}$, $d_i \in \mathbb{N}$, $c, \alpha, w, b_i \in \mathbb{C}$, and $|q/w| < |q^j| < |q^{n-1}/c|$. Together with (47), this can be used to find $q$-versions of almost all of the identities in sections 3 and 4. Results without $q$-versions at present include the generating function identity, the iterated contiguous relations, and several identities in section 2.

The proofs of the $q$-versions turn out to be routine, following the non-$q$-versions step by step, so we simply list the theorems for future reference, without proof. There is one exception; the details of the proof of the $q$-version of (43) are included, since a slightly different method of proof was used in the $q$-case then in the non-$q$-case. The proof of the $q$-version of (44) is similar to that of (43). We use the standard notation

$$
(w;q)_n := (w)_n := (1-w)(1-wq)\cdots(1-wq^{n-1}),
$$

$$
[x] := \frac{1-q^x}{1-q}, [x]_k := [x][x+1]\cdots[x+k-1],
$$

$$
\begin{bmatrix} x \\ k \end{bmatrix} := \frac{[x][x-1]\cdots[x-k+1]}{[k]!}, (z;q)_\infty := (z)_\infty := \prod_{k=0}^{\infty}(1-zq^k).
$$

In all of the following $q$-identities, $B$ is assumed to be a regular Ferrers board, except for equations (51),(52),(54), and (55), which hold for any Ferrers board. After listing the $q$-versions of previous results, we finish by deriving some new ones using transformations for $q$-series of all orders.

**Lemma 5.1** For any regular Ferrers board $B$,

$$
[k]!R_{n-k}(y) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j,y,B], \tag{49}
$$

and

$$
A_k(x,y,B) = \sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix} \begin{bmatrix} x+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} PR[j,y,B]. \tag{50}
$$

**Lemma 5.2** For any regular Ferrers board $B$,

$$
[k]!R_{n-k}(y) = PR[0,y,B](-1)^k q^{\binom{k}{2}} \quad {}_{t+1}\phi_t\left(\begin{matrix} q^{-k}, & q^{e_1+d_1}, & \ldots, & q^{e_t+d_t} \\ & q^{e_1}, & \ldots, & q^{e_t} \end{matrix}; q\right),
$$

and

$$A_k(x,y,B) = PR[0,y,B](-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+x \\ k \end{bmatrix}$$

$$\times {}_{t+2}\phi_{t+1}\left( \begin{matrix} q^{-k}, & q^{x}, & q^{e_1+d_1}, & \dots, & q^{e_t+d_t} \\ & q^{n+x-k+1}, & q^{e_1}, & \dots, & q^{e_t} \end{matrix} ; q \right).$$

**Lemma 5.3** For any Ferrers board,

$$A_k(x,1,B) = PR[u,1,B](-1)^{k-u} q^{\binom{k-u}{2}} \begin{bmatrix} n+x \\ k-u \end{bmatrix} \begin{bmatrix} x+u-1 \\ u \end{bmatrix}$$

$${}_{t+2}\phi_{t+1}\left( \begin{matrix} q^{-k+u}, & q^{x+u}, & q^{b_1+d_1}, & \dots, & q^{b_p+d_p}, & \dots, & q^{b_t+d_t} \\ q^{n+x-k+1+u}, & q^{u+1}, & q^{b_1}, & \dots, & \widehat{q^{b_p}}, & \dots, & q^{b_t} \end{matrix} ; q \right), \quad (51)$$

where $b_i := H_i - D_i + 1 + u$, $p$ is chosen so that $H_p - D_p = \min_{(i)} H_i - D_i$, and $u := D_p - H_p$.

**Theorem 5.4** For any regular Ferrers board $B$

$$A_k(x,y,B) = A_{n-k}(x, 1+x-y+n-H_t-p, \widehat{B}_p) q^{\alpha},$$

where $\alpha := n(-x+y-n) + k(n+x+1) +$ *area*$(B)$, with *area*$(B) =$ the number of squares in $B = \sum_i H_i d_i$. As in Theorem 3.1, $p$ is any positive integer for which $\widehat{B}_p$ is regular.

**Corollary 5.5** For any Ferrers board $B$

$$A_k(x,1,B) = A_{n-k}(x, x+n-H_t-p, \widehat{B}_p) q^{\beta}, \quad (52)$$

where $\beta := n(-x+1-n) + k(n+x-1) +$ *area*$(B)$.

**Lemma 5.6**

$${}_{t+1}\phi_t\left( \begin{matrix} q^{x}, & w_1 q^{d_1}, & \dots, & w_t q^{d_t} \\ & w_1, & \dots, & w_t \end{matrix} ; z \right) = \frac{(q^{n+x}z)_\infty}{(z)_\infty} \sum_{k=0}^{n} z^k \begin{bmatrix} n+x \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}}$$

$$\times {}_{t+2}\phi_{t+1}\left( \begin{matrix} q^{-k}, & q^{x}, & w_1 q^{d_1}, & \dots, & w_t q^{d_t} \\ & q^{n+x-k+1}, & w_1, & \dots, & w_t \end{matrix} ; q \right),$$

where $x, w_i, z \in \mathbb{C}$, $d_i \in \mathbb{N}$, and $n = D_t$. This gives a new derivation of the analytic continuation of the LHS to all of $\mathbb{C}$.

**Lemma 5.7** A $q$-version of (43);

$$A_k(x,y,B) = [k+e_p+d_p-1] A_k(x,y,B-h_p-d_p) +$$

$$q^{k+e_p+d_p-2}[n+x-e_p-d_p+1-k] A_{k-1}(x,y,B-h_p-d_p). \quad (53)$$

*Proof* : By Lemma 5.1,

$$A_k(x,y,B) = \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}} PR[s,y,B]$$

$$= \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}} [s+e_p+d_p-1] PR[s,y,B-h_p-d_p]$$

$$= \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}} PR[s,y,B-h_p-d_p]$$
$$\{[k+e_p+d_p-1] - q^{s+e_p+d_p-1}[k-s]\}$$

$$= [k+e_p+d_p-1] \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}} PR[s,y,B-h_p-d_p]$$

$$-q^{e_p+d_p-1} \sum_{s=0}^{k-1} [n+x] \begin{bmatrix} n+x-1 \\ k-1-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix} (-1)^{k-s} q^{\binom{k-s}{2}+s} PR[s,y,B-h_p-d_p]$$

$$= [k+e_p+d_p-1] \sum_{s=0}^{k} \{ \begin{bmatrix} n+x-1 \\ k-s \end{bmatrix} q^{\binom{k-s}{2}} + \begin{bmatrix} n+x-1 \\ k-1-s \end{bmatrix} q^{\binom{k-1-s}{2}+n+x-1} \} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}$$
$$\times PR[s,y,B-h_p-d_p] - q^{e_p+d_p-1} \sum_{s=0}^{k-1} [n+x] \begin{bmatrix} n+x-1 \\ k-1-s \end{bmatrix} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}$$
$$\times (-1)^{k-s} q^{\binom{k-1-s}{2}+k-1} PR[s,y,B-h_p-d_p]$$

(since $\left[\begin{smallmatrix} n+x \\ k-s \end{smallmatrix}\right] = \left[\begin{smallmatrix} n-1+x \\ k-s \end{smallmatrix}\right] + q^{n+x-k+s} \left[\begin{smallmatrix} n-1+x \\ k-1-s \end{smallmatrix}\right]$)

$$= [k+e_p+d_p-1] A_k(x,y,B-h_p-d_p) +$$
$$A_{k-1}(x,y,B-h_p-d_p)(-q^{n+x-1}[k+e_p+d_p-1] + q^{k+e_p+d_p-2}[n+x])$$

(by Lemma 5.1 applied with $B = B - h_p - d_p$). ■

**Theorem 5.8** Let $j \in \mathbb{N}$, $j \le h_p, d_p$ for some $p$ in the range $1 \le p \le t$. Then if $B$ is a regular Ferrers board, and $B_j$ is the board described in Theorem 4.1,

$$A_k(x,y,B) = [j]! \sum_{s=k-j}^{k} A_s(x,y,B_j) \begin{bmatrix} T_p-1+s \\ j-k+s \end{bmatrix} \begin{bmatrix} n-T_p+x-s \\ k-s \end{bmatrix} q^{(k-s)(T_p+k-j-1)},$$

where $A_s(x,y,\emptyset) = \delta_{s,0}$, and $T_p = H_p - D_{p-1} + y = e_p + d_p$.

**Theorem 5.9** Let $B$, $j$ and $B_{(j)}$ be as in Theorem 4.2. Then

$$A_k(x,y,B) =$$
$$[j]! \sum_{s=k-j}^{k} A_s(x,y,B_{(j)}) \begin{bmatrix} e_p+s+j-1 \\ j-k+s \end{bmatrix} \begin{bmatrix} n-e_p+x-s-j \\ k-s \end{bmatrix} q^{(k-s)(e_p+k-1)}.$$

**Corollary 5.10** Let $d_p, h_p, h_{p+1}, B$, and $B'$ be as in Corollary 4.3. Then

$$A_k(x,y,B)$$
$$= [d_p]! \sum_{s=k-d_p}^{k} A_s(x,y,B') \begin{bmatrix} e_p+d_p+s-1 \\ d_p-k+s \end{bmatrix} \begin{bmatrix} n-e_p-d_p+x-s \\ k-s \end{bmatrix} q^{(k-s)(e_p+k-1)}.$$

There are also $q$-versions of Theorems 2.2 and 2.3;

**Theorem 5.11** For any Ferrers board $B$, with $0 \le j \le n$, and $|z| < 1$,

$$\sum_{k=0}^{\infty} \begin{bmatrix} x+k-1 \\ k \end{bmatrix} A_j(-k,y,B) q^{jk} z^k = (-1)^j q^{\binom{j}{2}} \frac{(q^{j+x}z)_\infty}{(z)_\infty} \sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix} A_k(x,y,B) z^k. \tag{54}$$

**Theorem 5.12** For any Ferrers board $B$, with $0 \le j \le n$,

$$A_j(x,y,B) = q^{\binom{j}{2}} \sum_{k=0}^{j} \begin{bmatrix} n-k \\ n-j \end{bmatrix} A_k(y,y,B) \frac{(q^x)_k (q^{y-x})_{j-k}}{(q^y)_j} (-q^x)^{j-k} q^{-\binom{k}{2}}. \tag{55}$$

The following general expansion for a ${}_{t+1}\phi_t$ appears in [GaRa;p.110]; it is a special case of identities of Sears and Slater

$${}_{t+1}\phi_t \left[ \begin{matrix} w, & c_1, & \ldots, & c_t \\ & b_1, & \ldots, & b_t \end{matrix} ; z \right] = \frac{(wz)_\infty (q/wz)_\infty}{(z)_\infty (q/z)_\infty} \prod_{i+1}^{t} \frac{(c_i)_\infty (b_i/w)_\infty}{(b_i)_\infty (c_i/w)_\infty}$$

$$\times {}_{t+1}\phi_t \left[ \begin{matrix} w, & wq/b_1, & \ldots, & wq/b_t \\ & wq/c_1, & \ldots, & wq/c_t \end{matrix} ; \frac{qb_1 \cdots b_t}{zwc_1 \cdots c_t} \right] + \mathrm{idem}(w; c_1, \ldots, c_t), \tag{56}$$

where $|z| < 1$, $|\frac{qb_1 \cdots b_t}{zwc_1 \cdots c_t}| < 1$, and "idem$(w; c_1, \ldots, c_t)$" stands for the sum of the $t$ expressions obtained by interchanging $w$ and $c_1$ in the infinite products and ${}_{t+1}\phi_t$ on the RHS of (56), then interchanging $w$ and $c_2$, etc.

Making the following replacments in (56); $z = q^{1-j}/w$, $b_1 = cq\alpha$, $c_1 = \alpha$, $t = t+1$, $b_i = e_{i-1}, c_i = e_{i-1}q^{d_{i-1}}$ for $i \ge 2$, we get (48) after some simplification. If we make the same replacements but allow some of the $d_i$ to be negative integers, the RHS side of (56) reduces to a sum of terminating series. For example, if $d_1 = -5$, the term in the "idem" sum obtained by interchanging $w$ and $e_1 q^{-5}$ will have $qe_1 q^{-5}/e_1 = q^{-4}$ as one of the numerator parameters in the ${}_{t+1}\phi_t$ which will cause this series to terminate.

If we replace $c_i$ by $b_i q^{d_i}$ in (56) we get

$${}_{t+1}\phi_t \left( \begin{matrix} \alpha, & b_1 q^{d_1}, & \ldots, & b_t q^{d_t} \\ & b_1, & \ldots, & b_t \end{matrix} ; q, z \right) = \frac{(q/\alpha z)_\infty (\alpha z)_\infty}{(q/z)_\infty (z)_\infty}$$

$$\times \prod_{i=1}^{t} \frac{(b_i/\alpha)_{d_i}}{(b_i)_{d_i}} \quad {}_{t+1}\phi_t \left( \begin{matrix} \alpha, & q\alpha/b_1, & \ldots, & q\alpha/b_t \\ & q^{1-d_1}\alpha/b_1, & \ldots, & q^{1-d_t}\alpha/b_t \end{matrix} ; q, q^{1-n}/\alpha z \right)$$

valid for $\alpha, b_i, z \in \mathbb{C}$, $d_i \in \mathbb{N}$, with $n = D_t$. Letting $q$ approach 1 above and using elementary facts about the $q$-Gamma function [GaRa, Ch.1] yields the hypergeometric limit

$${}_{t+1}F_t \left[ \begin{matrix} x, & b_1+d_1, & \ldots, & b_t+d_t \\ & b_1, & \ldots, & b_t \end{matrix} ; z \right] =$$

$$(-z)^{-x} \prod_{i=1}^{t} \frac{(b_i - x)_{d_i}}{(b_i)_{d_i}} \quad {}_{t+1}F_t \left[ \begin{matrix} x, & 1+x-b_1, & \ldots, & 1+x-b_t \\ & 1+x-b_1-d_1, & \ldots, & 1+x-b_t-d_t \end{matrix} ; z^{-1} \right] \tag{57}$$

where $x, b_i, z \in \mathbb{C},\ d_i \in \mathbb{N},\ z \in \mathbb{C}\backslash[0,\infty)$, and the principal branch of $\log z$ is used to define $z^w$, as was used in Theorem 3.4. Letting $z=-1$ and $b_i=(1+x-d_i)/2$ proves Theorem 3.5 when the argument in (34) is -1.

Another useful expansion for the general ${}_{t+1}\phi_t$ is the corollary of Bowman's generalized Heine discussed in section 1. Specializing the RHS of (18) to the Karlsson-Minton case by setting $c_i = b_i q^{d_i}$ we get

$$ {}_{t+1}\phi_t \left[\begin{matrix} x, & b_1q^{d_1}, & \dots, & b_tq^{d_t} \\ & b_1, & \dots, & b_t \end{matrix} ; z\right] = \frac{(xz)_\infty}{(z)_\infty} \prod_{i=1}^{t} \frac{1}{(b_i)_{d_i}} $$

$$ \times \sum_{k=0}^{\infty} \frac{(z)_k}{(xz)_k} \sum_{m_1+\ldots+m_t=k} (-1)^k \prod_{i=1}^{t} \begin{bmatrix} d_i \\ m_i \end{bmatrix} b_i^{m_i} q^{\binom{m_i}{2}}. \tag{58} $$

To translate this into a result on rook polynomials, note that the $j=n$ case of Theorem 5.11 is equivalent to (using (47))

$$ \sum_{k=0}^{\infty} \begin{bmatrix} x+k-1 \\ k \end{bmatrix} PR[k,y,B] z^k = \frac{(q^{n+x}z)_\infty}{(z)_\infty} \sum_{k=0}^{n} z^k A_k(x,y,B), $$

which reduces to an identity of Garsia and Remmel when $x=y=1$. Now the LHS above equals

$$ PR[0,y,B]_{t+1}\phi_t \left[\begin{matrix} x, & q^{e_1+d_1}, & \dots, & q^{e_t+d_t} \\ & q^{e_1}, & \dots, & q^{e_t} \end{matrix} ; z\right], $$

so the $b_i = q^{e_i}, x = q^x$ case of (58) implies

**Corollary 5.13** For any Ferrers board $B$

$$ \sum_{k=0}^{n} z^k A_k(x,y,B) = $$

$$ \frac{1}{(1-q)^n} \sum_{k=0}^{n} (z)_k (q^{x+k}z)_{n-k} (-1)^k \sum_{m_1+\ldots+m_t=k} \prod_{i=1}^{t} \begin{bmatrix} d_i \\ m_i \end{bmatrix} q^{e_i m_i + \binom{m_i}{2}}. $$

The special case $e_i \equiv 1, d_i \equiv 1, x=1$ gives

$$ \sum_{k=0}^{n} z^k \sum_{\substack{\sigma\in S_n \\ k\ descents}} q^{maj\sigma} = \frac{(z)_{n+1}}{(1-q)^n} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} q^k}{1-zq^k}, $$

where the inner sum on the LHS is over all permutations $\sigma_1\sigma_2\cdots\sigma_n$ of $S_n$ having $k$ values of $i$ for which $\sigma_i > \sigma_{i+1}$, and $maj\sigma := \sum_{i:\sigma_i>\sigma_{i+1}} i$. Letting $z=1$ in Corollary 5.13 we get the simple identity

$$ \sum_{k=0}^{n} A_k(x,y,B) = [x]_n. $$

The result

$$ \sum_{k=0}^{n} R_k (1-q)^k = 1, $$

valid for any Ferrers board B, can also be derived from 5.13. If $B$ is the "$m$-jump board", $B = B(m,1;m,1;\dots;m,1)$, then 5.13 implies

$$R_{n-k}(B)(1-q)^{n-k} = \sum_{j\geq 0}(-1)^{j+k}\begin{bmatrix} j \\ k \end{bmatrix} q^{\binom{k}{2}+mj+(m-1)\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{m-1}},$$

where the last $q$-binomial coefficient is in the base $q^{m-1}$. Using (28), this identity can be shown to reduce to a special case of Fine's bibasic version of the Heine transformation [Fine, Eq. (21.81)].

## 6. Summary.

The subject of hypergeometric series has been the focus of a great deal of research over the past 200 years. Many of these results, especially theorems on series of Karlsson-Minton type, have application to the study of rook polynomials. As this young branch of enumerative combinatorics matures, perhaps the combinatorial perspective it provides will result in new developments in hypergeometric series.

**Acknowledgements.** The author would like to thank Professors Andrews, Bowman, and Stanton for supplying references to some of the theorems on hypergeometric series used in this article. He would also like to thank the referee for helpful comments including a formula (Lemma 2.4) which led to Theorem 2.7.

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