Homework Set 3 (Due in class on Thursday, Oct. 1)

(late papers accepted until 1:00 Friday)

The problem numbers refer to the D'Angelo - West text.

1. a) If a and b are rational numbers, show that the set S of real numbers of the form $a + b\sqrt{7}$ form a field. Note that since S is a subset of the real numbers, it automatically inherits many of the properties of a field. Consequently, you need only show the following:
   i) If z and w are in S, then so are $z + w$ and zw. So the set S is *closed* under both addition and multiplication.
   ii) The elements in S have additive inverses in S.
   iii) The elements in S have multiplicative inverses in S.

   b) Does the set T of real numbers of the form $a + b\sqrt{6}$ form a field? (Again, a and b are rational numbers).

2. If $x > 0$, use induction to verify the inequality $(1 + x)^k \geq 1 + kx$ for any integer $k = 1, 2, \ldots$.

3. A standard example of using induction is to verify the formula $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. It is much less well-known that you can use induction to *discover* this formula. Here is the procedure. Let $S_n := 1+2+3+\cdots+n$. Then the key (induction) step is

$$S_{n+1} - S_n = n + 1 \quad \text{(why?)} \qquad \text{with the } \textit{initial condition} \quad S_1 = 1. \tag{1}$$

   We want to solve this *difference equation* (1) directly. Experimenting, we find $S_2 = 3$, $S_3 = 6$, $S_4 = 10$, $S_5 = 15$. This leads us to guess there might be a formula of the form

$$S_n = an^2 + bn + c,$$

   where the coefficients a, b, and c are still unknown. To find them we plug this into (1) and hope we will succeed in finding them. Do this. You will succeed.

   It could be that equation (1) has several solutions. But it doesn't. Show that if $\hat{S}_n$ also satisfies (1), then $\hat{S}_n = S_n$. [MORAL: If you have found a solution, then you have found the unique solution.]

4. Use the idea of the previous problem to find a formula for $T_n := 1^2 + 2^2 + 3^2 + \cdots + n^2$.

5. Let a_n be a sequence of real numbers with the property that the "even" subsequence a_{2n} converges to A and the "odd" subsequence a_{2n+1} converges to B [Baby Example: $a_n = (-1)^n$]. If the original sequence a_n converges, show that $A = B$ (so the baby example does **not** converge).

6. [# 13.9] Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \dfrac{2x - 8}{x^2 - 8x + 17}$. Then the supremum of the image of f is 1. Give a proof or counterexample.

7. Show that $\lim_{n\to\infty} \dfrac{10^n}{n!} = 0$.

8. [# 13.34]

   a) Given any two rational numbers $r < s$, prove there is an *irrational* number c between them: $r < c < s$.

   b) Given any two real numbers $a < b$, prove there is a *rational* number r between them: $a < r < b$.

9. [#13.11] Suppose the sequences a_n and b_n of real numbers both converge. For each of the following assertions give a proof or counterexample,

   a) If $\lim a_n < \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n < b_n$.

   b) If $\lim a_n \leq \lim b_n$, then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $a_n \leq b_n$.

[Last revised: October 2, 2009]