Making a Loan

Melinda borrows P_0 dollars to buy a house. The annual interest rate is i (perhaps 6.5%) compounded monthly. Thus the monthly interest rate is $i/12$, so at the end of the first month she owes $[1+(i/12)]P_0$ dollars. Note that if the annual interest rate is 6.5%, then i is written as, .065. She will repay the loan with equal monthly payments of M dollars. Thus, just after she has made the first monthly payment the new principal she owes is

$$P_1 = \left(1+\frac{i}{12}\right)P_0 - M \qquad \text{and} \qquad P_k = \left(1+\frac{i}{12}\right)P_{k-1} - M \tag{1}$$

dollars.

What is the minimum monthly payment?

It is $(i/12)P_0$ since only then will the balance decrease: $P_1 < P_0$. Thus we require that $M > \frac{i}{12}P_0$.

What is the unpaid balance after k monthly payments?

At the end of two months she owes $\left(1+\frac{i}{12}\right)P_1$ dollars so just after she has made the second monthly payment she owes $P_2 = \left(1+\frac{i}{12}\right)P_1 - M$ and so on. To clean the computations, we let $c = 1+\frac{i}{12}$. Then using the formula for P_1 we have

$$P_2 = c^2P_0 - (1+c)M, \quad \text{and similarly,} \quad P_3 = c^3P_0 - (1+c+c^2)M.$$

Continuing we conclude that just after making the k^{th} monthly payment she owes

$$P_k = c^kP_0 - (1+c+c^2+\cdots+c^{k-1})M. \tag{2}$$

Using the standard formula for the sum of a geometric series, this becomes

$$P_k = c^kP_0 - \frac{c^k-1}{c-1}M. \tag{3}$$

How many monthly payments, N, are needed until the loan is completely repaid?

This means given M, find N so that $P_N = 0$. It will be convenient to let $j = c-1 = \frac{i}{12}$, which is the monthly interest rate. From (3) with $k = N$:

$$\begin{aligned} 0 &= c^NP_0 - \frac{c^N-1}{c-1}M & (4)\\ &= c^N\left(P_0 - \frac{M}{j}\right) + \frac{M}{j}, & (5) \end{aligned}$$

so

$$c^N = \frac{M}{M - jP}.$$

Recall that $M - jP > 0$, as we saw above when calculating the minimum monthly payment. We take logarithms to solve for N:

$$N = \frac{\log\left(\frac{M}{M-jP}\right)}{\log c} = \frac{\log\left(\frac{M}{M-jP}\right)}{\log(1+j)}. \tag{6}$$

N will rarely be an integer. All this means is that the last payment will be smaller, just enough to cover the balance.

If the loan is to be repaid in exactly N monthly payments, how much, M, should she repay each month?

This means given N, find M so that $P_N = 0$. Thus we solve (4) for M and obtain

$$M = \frac{c^N(c-1)P_0}{c^N - 1}. \tag{7}$$

As a brief check, note that if $N = 1$ we get $M = cP_0$, as expected.

In the k^{th} payment of M dollars, how much is interest and how much goes to reducing the principal?

Rewrite equation (1) as

$$M = [P_0 - P_1] + [\frac{i}{12}P_0].$$

This shows how the first payment of M dollars is allocated: $P_0 - P_1$ is the amount the balance is decreased while $\frac{i}{12}P_0$ is the interest paid on the balance. Similarly, using (1) we find

$$M = [P_{k-1} - P_k] + \frac{i}{12}P_{k-1},$$

which exhibits the k^{th} payment as reducing the balance by $P_{k-1} - P_k$ while the interest paid that month is $\frac{i}{12}P_{k-1}$. Using (3) for P_{k-1} we conclude that

$$\begin{aligned}
\text{Interest paid in } k^{\text{th}} \text{ month} = \frac{i}{12}P_{k-1} &= \frac{i}{12}\left(c^{k-1}P_0 - \frac{c^{k-1}-1}{c-1}M\right) && (8)\\
&= \frac{i}{12}c^{k-1}P_0 - (c^{k-1}-1)M. && (9)
\end{aligned}$$