A critical analysis of the eigenvalue method used to derive priorities in AHP

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Abstract
A lot of research has been devoted to the critical analysis of the Analytic Hierarchy Process (AHP), from various perspectives. However, as far as we know, no one has addressed a fundamental problem, discussed in this paper, concerning the meaning of the priority vector derived from the principal eigenvalue method used in AHP. The role of AHP's consistency ratio is also analysed.
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1. Introduction and objective of the analysis

Since Thomas L. Saaty (1977, 1980) introduced the *Analytic Hierarchy Process* (AHP), many applications in real-world decision-making have been reported (*cf.* Zahedi, 1986; Golden *et al.*, 1989; Shim, 1989; Vargas, 1990, Saaty, 2000, Forman and Gass, 2001, Golden and Wasil, 2003, Vaidya and Kumar, 2006). In parallel, AHP has often been criticised in the literature, from several perspectives (see, for example, Watson and Freeling, 1982 and 1983; Belton and Gear, 1983 and 1985; French, 1988; Holder, 1990; Dyer, 1990a and b; Barlizai and Golany, 1994; Salo and Hämäläinen, 1997). A debate about the main criticisms of AHP can be found in (Belton and Stewart, 2002) and (Smith and von Winterfeldt, 2004). Saaty has frequently contested these critics (see, for example, Saaty *et al.*, 1983; Saaty and Vargas, 1984; Saaty, 1990 and 1997; Saaty and Hu, 1998) and, in essence, has not modified his original method (see Saaty, 2005). Independently of our agreement with some of those criticisms, the analysis of which is beyond the scope of this paper, we believe that the elicitation of pairwise comparison judgements and the possibility of expressing them verbally are cornerstones of the popularity of AHP.

There is, however, a key problem that, as far as we know, has never before been addressed in the literature. It concerns the meaning of the priority vector derived from the principal eigenvalue method used in AHP. The "AHP uses a principal Eigenvalue Method (EM) to derive priority vectors" (Saaty and Hu, 1998, p. 121). Following Saaty, the priority vector has two meanings: "The first is a numerical ranking of the alternatives that indicates an order of preference among them. The other is that the ordering should also reflect intensity or cardinal preference as indicated by the ratios of the numerical values (…)"

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(Saaty, 2003, p. 86). This second meaning requires, in our view, that these ratios preserve, whenever possible, the order of the respective preference intensities, which is not always the case for AHP priority vectors. Indeed, the ratios of AHP priority values can violate this order albeit the ratios of alternative priority values, derived from the same pairwise comparisions, preserve it. From our decision-aid perspective, this is a basic drawback of AHP. Consider the following condition:

__C__ondition of __O__rder __P__reservation (COP): For all alternatives x_1, x_2, x_3, x_4 such that x_1 dominates[1] x_2 and x_3 dominates x_4, if the evaluator's judgements indicate the extent to which x_1 dominates x_2 is greater that the extent to which x_3 dominates x_4, then the vector of priorities w should be such that, not only $w(x_1) > w(x_2)$ and $w(x_3) > w(x_4)$ (preservation of order of preference) but also that $w(x_1)/w(x_2) > w(x_3)/w(x_4)$ (preservation of order of intensity of preference).

For instance, if x_1 strongly dominates x_2 and x_3 moderately dominates x_4, it is from our view fundamental that, whenever possible, the vector of priorities w be such that $w(x_1)/w(x_2) > w(x_3)/w(x_4)$; indeed, these judgements indicate that the intensity of preference of x_1 over x_2 is higher than the intensity of preference of x_3 over x_4.

We will prove with simple examples that the AHP priority vector does not necessarily satisfy the COP, even though it is possible to respect this condition. In such cases, alternative priority values that satisfy COP can easily be found by a mathematical program including COP constrains. The particular program that we used is not important in the scope of this paper, since our intention is not at all to suggest an alternative procedure to AHP.

Note that a numerical scale that satisfies the COP does not always exist. In our constructive perspective, it is essential to detect these situations and discuss them with the evaluator before proposing any priority scale. A complementary objective of this paper is to analyse if the consistency ratio used in AHP can reveal such situations.

The rest of this paper is organised in the following manner: in section 2, we review the principal eigenvalue method used in AHP to derive priority vectors; in sections 3 and 4, we present some examples in which it would be possible to satisfy the COP, however, the AHP priority vectors violate it; in section 5, we show that the AHP consistency ratio is not suitable for detecting the existence (or the non existence) of a numerical scale satisfying the COP; a brief conclusion is presented in section 6.

2. Overview of the principal eigenvalue method (EM)

Let $X = \{x_1, x_2, ..., x_n\}$ be a set of elements and $\wp$ "a property or criterion that they have in common" (Saaty, 1996, p. 24) – for example, X could be a set of cars and $\wp$ their comfort. How can we help a person J quantify the relative priority (or importance) that the elements of X have for her, in terms of $\wp$?

The EM used in AHP to derive priorities for the elements of X requires that a number – denoted w_{ij} – be assigned to each pair of elements (x_i, x_j)

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1 In this paper, "dominance" is used in the sense of "strict preference".

representing, in the opinion of J, the ratio of the priority of the dominant element (x_i) relative to the priority of the dominated element (x_j) (Saaty, 1997). J is invited to compare the elements pairwise and can express her judgements in two different ways:

- either numerically, by giving a real number between 1 (inclusive) and 10 (exclusive) (Saaty, 1989) – for example, if x_i is a Chevrolet and x_j a Lada and if J judges the Chevrolet to be six times more comfortable than the Lada, than $w_{ij} = 6$.
- or verbally, by choosing one of the following expressions: equal importance, moderate dominance, strong dominance, very strong dominance, extreme dominance, or an intermediate judgement between two consecutive expressions; each verbal pairwise comparison elicited is then automatically converted into a number w_{ij} as exhibited in Table 1 – for example, if x_i is a Peugeot and x_j an Opel and if J judges the Peugeot to be moderately more comfortable than the Opel, then $w_{ij} = 3$.

Table 1: Converting "verbal judgements" into "numbers".

Verbal expressions[2]	Corresponding numbers
Equal	1
equal to moderate	2
moderate	3
moderate to strong	4
Strong	5
strong to very strong	6
very strong	7
very strong to extreme	8
extreme	9

During the elicitation process, a positive reciprocal matrix, in which each element x_1, x_2, ..., x_n of X is assigned one line and one column, can be filled by placing the corresponding number at the intersection of the line of x_i with the column of x_j

$$\begin{cases} w_{ij} & \text{if } x_i \text{ dominates } x_j \\ 1/w_{ij} & \text{if } x_j \text{ dominates } x_i \\ 1 & \text{if } x_i \text{ does not dominate } x_j \text{ and } x_j \text{ does not dominate } x_i \end{cases}$$

For example, assuming that for all i, j $\in$ {1, 2, ..., n} x_i dominates x_j if and only if $i < j$, the format of the positive reciprocal matrix will be

2 In (Saaty, 1996 and 2005) the verbal expressions "equal to moderate", "moderate to strong", "strong to very strong" and "very strong to extreme" are replaced by "weak", "moderate plus", "strong plus" and "very, very strong", respectively.

$$\mathbf{W} = \begin{pmatrix} 1 & w_{12} & \dots & w_{1n} \\ 1/w_{12} & 1 & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1/w_{1n} & 1/w_{2n} & \dots & 1 \end{pmatrix}$$

In order to assign a "priority" (or a "weight") to each element x_i – a numerical value that we will denote $w(x_i)$ – the principal eigenvalue λ_{max} of matrix **W** and its normalised eigenvector are calculated: the components of this vector are the $w(x_i)$. This procedure has a very interesting property: if the judgements of J are such that $w_{ij}.w_{jk} = w_{ik}$ for all i < j < k (cardinal consistency condition), the derived $w(x_i)$ are such that $w_{ij} = w(x_i)/w(x_j)$ for all i < j.

However, cardinal consistency is seldom observed in practice. Therefore, AHP makes use of a "consistency test" that prevents priorities from being accepted if the inconsistency level is high. In order to measure the deviation of matrix **W** from "consistency", a consistency index C.I. is defined as λ_{max}-n/(n-1) and a random index R.I. (of order n) is calculated as the average of the C.I. of many thousands reciprocal matrices (of order n) randomly generated from the scale 1 to 9, with reciprocals forced. The values of R.I. for matrices of size 1, 2, …, 10 can be found in (Saaty, 2005, p. 374). The ratio of C.I. to R.I. for the same order matrix is called the consistency ratio C.R.. According to (Saaty, 1980, p. 21), "a consistency ratio of 0.10 or less is considered acceptable". That is, an inconsistency is stated to be a matter of concern if C.R. exceeds 0.1, in which case the pairwise comparisons should be re-examined.

If the elements are to be compared according to several $\wp$, the AHP proposes that a hierarchy be built with the general goal on top, the elements at the bottom and the $\wp$ at intermediate levels. The procedure described above is then repeatedly applied bottom-up: to calculate a vector of priorities for the elements with respect to each $\wp$ situated at the bottom intermediate level; to calculate a vector of weights for each cluster of $\wp$ at the different levels. All this judgmental information is then synthesised from bottom to top by successive additive aggregations, in order to derive a vector of overall priorities for the elements.

3. Examples in which the COP is violated by the priority vector derived from the EM

We present in this section two examples proving that the COP may be violated by the priority vector given by the EM for each one of them, although scales exist that do respect it. Example 1 involves verbal judgements and example 2 involves numerical judgements.

<u>Example 1 (case of verbal judgements)</u>

Let X = {x_1, x_2, x_3, x_4, x_5} be a set of alternatives between which the following pairwise comparisons were formulated by a person J:

$\{x_1, x_2\}$: x_1 dominates x_2, equal to moderate dominance
$\{x_1, x_3\}$: x_1 dominates x_3, moderate dominance
$\{x_1, x_4\}$: x_1 dominates x_4, strong dominance
$\{x_1, x_5\}$: x_1 dominates x_5, extreme dominance
$\{x_2, x_3\}$: x_2 dominates x_3, equal to moderate dominance
$\{x_2, x_4\}$: x_2 dominates x_4, moderate to strong dominance
$\{x_2, x_5\}$: x_2 dominates x_5, extreme dominance
$\{x_3, x_4\}$: x_3 dominates x_4, equal to moderate dominance
$\{x_3, x_5\}$: x_3 dominates x_5, very strong to extreme dominance
$\{x_4, x_5\}$: x_4 dominates x_5, very strong dominance.

From Table 1, the corresponding positive reciprocal matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 9 \\ 1/2 & 1 & 2 & 4 & 9 \\ 1/3 & 1/2 & 1 & 2 & 8 \\ 1/5 & 1/4 & 1/2 & 1 & 7 \\ 1/9 & 1/9 & 1/8 & 1/7 & 1 \end{pmatrix}$$

for which the normalised eigenvector corresponding to its principal eigenvalue is

$$\begin{pmatrix} 0.426 \\ 0.281 \\ 0.165 \\ 0.101 \\ 0.027 \end{pmatrix}.$$

Consequently, given the judgements of J, the priorities obtained through the EM are

$$w(x_1) = 0.426$$
$$w(x_2) = 0.281$$
$$w(x_3) = 0.165$$
$$w(x_4) = 0.101$$
$$w(x_5) = 0.027.$$

Then, in particular, $w(x_1)/w(x_4) \approx 4.218$ and $w(x_4)/w(x_5) \approx 3.741$, that is, $w(x_1)/w(x_4) > w(x_4)/w(x_5)$. Given that J judged that x_4 very strongly dominates x_5 and x_1 strongly dominates x_4, the priority vector given by the EM violates the COP. Yet, for example, the scale w*

$$w^*(x_1) = 0.385$$
$$w^*(x_2) = 0.275$$
$$w^*(x_3) = 0.195$$
$$w^*(x_4) = 0.125$$
$$w^*(x_5) = 0.020$$

respects the COP, as shown in Table 2. Let us also point out that the value of the consistency ratio for the judgements in example 1 is 0.05, significantly smaller than the 0.10 threshold; therefore in AHP's perspective the judgements need not be revised.

Table 2: Example 1 – values of the ratios $w^*(x_i)/w^*(x_j)$.

Possible verbal judgements	(x_i,x_j) pair(s) and respective $w^*(x_i)/w^*(x_j)$ ratios
equal to moderate	(x_1,x_2): 1,40 (x_2,x_3): 1,41 (x_3,x_4): 1,56
Moderate	(x_1,x_3): 1,97
Moderate to strong	(x_2,x_4): 2,20
Strong	(x_1,x_4): 3,08
strong to very strong	$\varnothing$
very strong	(x_4,x_5): 6,25
Very strong to extreme	(x_3,x_5): 9,75
Extreme	(x_2,x_5): 13,75 (x_1,x_5): 19,25

Example 2 (case of numerical judgements)

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives between which the following pairwise comparisons were formulated by a person J:

$\{x_1, x_2\}$: x_1 dominates x_2 2.5 times
$\{x_1, x_3\}$: x_1 dominates x_3 4 times
$\{x_1, x_4\}$: x_1 dominates x_4 9.5 times
$\{x_2, x_3\}$: x_2 dominates x_3 3 times
$\{x_2, x_4\}$: x_2 dominates x_4 6.5 times
$\{x_3, x_4\}$: x_3 dominates x_4 5 times.

The corresponding positive reciprocal matrix is

$$\begin{pmatrix} 1 & 2.5 & 4 & 9.5 \\ 1/2.5 & 1 & 3 & 6.5 \\ 1/4 & 1/3 & 1 & 5 \\ 1/9.5 & 1/6.5 & 1/5 & 1 \end{pmatrix}$$

for which the normalised eigenvector corresponding to its maximal eigenvalue is

$$\begin{pmatrix} 0.533 \\ 0.287 \\ 0.139 \\ 0.041 \end{pmatrix}.$$

Consequently, given the judgements of J, the priorities obtained through the EM are

$$w(x_1) = 0.533$$
$$w(x_2) = 0.287$$
$$w(x_3) = 0.139$$
$$w(x_4) = 0.041.$$

For all $i,j \in \{1,2,3,4\}$ such that $i < j$, Table 3 presents the numerical value w_{ij} given by J when she judged how many times x_i dominates x_j, together with the respective value of the ratio $w(x_i)/w(x_j)$.

Table 3: Example 2 – values of w_{ij} and $w(x_i)/w(x_j)$.

	w_{ij}	$w(x_i)/w(x_j)$
$\{x_1, x_4\}$	9.5	13
$\{x_2, x_4\}$	6.5	7
$\{x_3, x_4\}$	5	3.39
$\{x_1, x_3\}$	4	3.83
$\{x_2, x_3\}$	3	2.06
$\{x_1, x_2\}$	2.5	1.86

It is not surprising that the values of $w(x_i)/w(x_j)$ are not the same as the numerical judgements w_{ij} (because the latter are not cardinally consistent) but it is surprising to verify that their order is not preserved by the ratios. Indeed, $w_{34} > w_{13}$ but $w(x_3)/w(x_4) < w(x_1)/w(x_3)$. This proves that, again, the priority vector given by the EM violates the COP. Yet, for example, the scale w*

$$
\begin{gathered}
w^*(x_1) = 0.48 \\
w^*(x_2) = 0.32 \\
w^*(x_3) = 0.16 \\
w^*(x_4) = 0.04
\end{gathered}
$$

respects the COP. Indeed,

$$
\frac{w^*(x_1)}{w^*(x_4)} = 12 > \frac{w^*(x_2)}{w^*(x_4)} = 8 > \frac{w^*(x_3)}{w^*(x_4)} = 4 > \frac{w^*(x_1)}{w^*(x_3)} = 3
$$
$$
> \frac{w^*(x_2)}{w^*(x_3)} = 2 > \frac{w^*(x_1)}{w^*(x_2)} = 1.5.
$$

Moreover, the value of the consistency ratio for the judgements in example 2 is 0.05, significantly smaller than the 0.10 threshold; therefore in AHP's perspective the judgements need not be revised.

4. Analysis of one of Saaty's examples

Example 3. In this section we analyse the violation of the COP in one of the examples presented in (Saaty, 1977, pp. 254-256) and (Saaty, 1980, pp. 40-41) to empirically validate the EM. We refer to the example of pairwise comparisons of the GNP of several countries, in which, for a given matrix of verbal judgements, the priorities given by the AHP are remarkably close to the normalised GNP values. The countries are (Saaty's notation) "U.S., U.S.S.R., China, France, U.K., Japan and W. Germany" and the matrix of judgements presented is

$$
\begin{array}{c}
\\ U.S. \\ U.S.S.R. \\ China \\ France \\ U.K. \\ Japan \\ W.Germany
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} U.S. & U.S.S.R. & China & France & U.K. & Japan & W.Germany \end{array} \\
\begin{pmatrix}
1 & 4 & 9 & 6 & 6 & 5 & 5 \\
1/4 & 1 & 7 & 5 & 5 & 3 & 4 \\
1/9 & 1/7 & 1 & 1/5 & 1/5 & 1/7 & 1/5 \\
1/6 & 1/5 & 5 & 1 & 1 & 1/3 & 1/3 \\
1/6 & 1/5 & 5 & 1 & 1 & 1/3 & 1/3 \\
1/5 & 1/3 & 7 & 3 & 3 & 1 & 2 \\
1/5 & 1/4 & 5 & 3 & 3 & 1/2 & 1
\end{pmatrix}
\end{array}
$$

The corresponding priorities are

w(U.S.) = 0.427
w(U.S.S.R.) = 0.230
w(China) = 0.021
w(France) = 0.052
w(U.K.) = 0.052
w(Japan) = 0.123
w(W. Germany) = 0.094.

These are the priorities appearing in (Saaty, 1980), which are a little different from those in (Saaty, 1977): 0.429, 0.231, 0.021, 0.053, 0.053, 0.119, and 0.095, respectively. Nevertheless, in both of these priority vectors the same five violations of the COP can be observed. We will analyse two of these hereafter.

1) According to the matrix of judgements, U.S. dominates U.S.S.R. (4 times) more than Japan dominates France (3 times). But, w(U.S.)/w(U.S.S.R.) ≈ 1.857 and w(Japan)/w(France) ≈ 2.365, that is, w(U.S.)/w(U.S.S.R.) < w(Japan)/w(France).

2) According to the matrix of judgements, Japan dominates China (7 times) more than U.S. dominates U.K. (6 times). But, w(Japan)/w(China) ≈ 5.857 and w(U.S.)/w(U.K.) ≈ 8.212, that is, w(Japan)/w(China) < w(U.S.)/w(U.K.).

In spite of this, it is possible to avoid all of the violations of the COP, as for example with the following priority vector of priorities w* (see Table 4):

w*(U.S.) = 0.414
w*(U.S.S.R.) = 0.217
w*(China) = 0.019
w*(France) = 0.069
w*(U.K.) = 0.069
w*(Japan) = 0.117
w*(W. Germany) = 0.095.

Table 4: Verification of the COP.

Possible verbal judgements	(x_i,x_j) pair(s) and respective $w^*(x_i)/w^*(x_j)$ ratios
equal to moderate	(Japan, W. Germany): 1.23
Moderate	(W. Germany, France): 1.38 (W. Germany, U.K.): 1.38 (Japan, France): 1,70 (Japan, U.K.): 1.70 (U.S.S.R., Japan): 1,85
moderate to strong	(U.S., U.S.S.R.): 1.91 (U.S.S.R., W. Germany): 2.28
Strong	(U.S.S.R., France): 3.14 (U.S.S.R., U.K.): 3.14 (U.S., Japan): 3.54 (U.K., China): 3,63 (France, China): 3.63 (U.S., W. Germany): 4.36
strong to very strong	(U.S., France): 6.00 (U.S., U.K.): 6.00
Very strong	(Japan, China): 6.16 (U.S.S.R., China): 11.42
very strong to extreme	$\varnothing$
Extreme	(U.S., China): 21.79

Let us also point out that the value of the consistency ratio for the judgements of this example is 0.08.

5. Discussion about the consistency ratio (C.R.)

<u>Example 4</u>: In this section we present an example in which it is impossible to find a numerical scale satisfying the COP and analyse the value of the C.R. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of alternatives between which the following pairwise comparison judgements were formulated by a person J:

$\{x_1, x_2\}$: x_1 dominates x_2, equal to moderate dominance
$\{x_1, x_3\}$: x_1 dominates x_3, strong dominance
$\{x_1, x_4\}$: x_1 dominates x_4, very strong dominance
$\{x_1, x_5\}$: x_1 dominates x_5, extreme dominance
$\{x_2, x_3\}$: x_2 dominates x_3, equal to moderate dominance
$\{x_2, x_4\}$: x_2 dominates x_4, moderate dominance
$\{x_2, x_5\}$: x_2 dominates x_5, very strong dominance
$\{x_3, x_4\}$: x_3 dominates x_4, moderate dominance
$\{x_3, x_5\}$: x_3 dominates x_5, strong dominance
$\{x_4, x_5\}$: x_4 dominates x_5, equal to moderate dominance.

For this set of judgements, it is impossible to satisfy the COP. Indeed, one should simultaneously have:

1) $w(x_1)/w(x_3) > w(x_2)/w(x_4)$, because, according to J's judgements, x_1 dominates x_3 (strong dominance) more than x_2 dominates x_4 (moderate dominance), and

2) $w(x_3)/w(x_4) > w(x_1)/w(x_2)$, because, according to J's judgements, x_3 dominates x_4 (moderate dominance) more than x_1 dominates x_2 (equal to moderate dominance).

This is impossible because the product, member to member, of these two inequalities gives $w(x_1)/ w(x_4) > w(x_1)/ w(x_4)$.

In our view, this shows that we are in face of a real case of judgemental inconsistency because, contrary to examples 1 to 3, the set of judgments in the present example is incompatible with a numerical representation that guarantees order preservation. And yet, the value of the C.R. corresponding to these judgements is very small (0.03), which means, in the AHP's perspective, that these judgements would not necessitate to be revised. Moreover, 0.03 is smaller than the values of the consistency ratios for examples 1 to 3 (0.05, 0.05 and 0.08) in which, as shown in sections 3 and 4, scales exist that satisfy the COP, unlike to the present example in which an inconsistency problem undoubtedly exists. This shows that the C.R. used in AHP is not suitable for detecting the existence (or the non existence) of a numerical scale satisfying the COP.

6. Conclusion

In this article, we have addressed the foundations of AHP, by analysing the eigenvalue method (EM) used to derive a priority vector. Our main conclusion is that, although the EM is very elegant from a mathematical viewpoint, the priority vector derived from it can violate a condition of order preservation that, in our opinion, is fundamental in decision aiding – an activity in which it is essential to respect values and judgements. In light of that, and independently of all other criticisms presented in the literature, we consider that the EM has a serious fundamental weakness that makes the use of AHP as a decision support tool very problematic. As Saaty (2005, p. 346) points out, "the purpose of decision-making is to help people make decisions according to their own understanding", and "… methods offered to help make better decisions should be closer to being descriptive and considerably transparent."

Finally, it is worthwhile to note that the criticism of the EM, presented in this paper, is also valid for any other method that has been (or may be) conceived to derive a vector of priorities from a pairwise comparison matrix on the basis of a mathematical technique that does not integrate what we call the COP, or does not automatically guarantee its satisfaction.

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