Exercise: Rotations in $\mathbb{R}^3$

Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector. The point of this problem is to outline the following:
Find a formula for the rotation of $\mathbb{R}^3$ through an angle θ with the direction $\mathbf{n}$ as axis of rotation. Before reading further, try finding it on your own.

a) (Example) Find a matrix that rotates $\mathbb{R}^3$ through the angle θ using the vector $(1,0,0)$ as the axis of rotation.

b) If $\mathbf{x} \in \mathbb{R}^3$ is any vector, show that $\mathbf{u} := \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$ is the projection of $\mathbf{x}$ is the projection of $\mathbf{x}$ into the plane perpendicular to $\mathbf{n}$. [Here $(\mathbf{x} \cdot \mathbf{n})$ is the usual dot product.]

c) Show that the vector

$$\mathbf{w} := \mathbf{u} \times \mathbf{n} = (\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}) \times \mathbf{n} = \mathbf{x} \times \mathbf{n}$$

is perpendicular to both $\mathbf{n}$ and $\mathbf{u}$, and that $\mathbf{w}$ has the same length as $\mathbf{u}$. Thus $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{w}$ are orthogonal with $\mathbf{u}$, and $\mathbf{w}$ in the plane perpendicular to the axis of rotation $\mathbf{n}$. [Here, $\mathbf{u} \times \mathbf{n}$ is the standard cross product of vectors in $\mathbb{R}^3$.]

d) Show that the map

$$R : \mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n} + \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{w}$$

rotates $\mathbf{x}$ through an angle θ with $\mathbf{n}$ as axis of rotation. [Note: one needs more information to be able to distinguish between θ and $-\theta$].

e) Show that one can rewrite the previous formula as

$$\begin{aligned} R\mathbf{x} &= (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n} + \cos\theta\,(\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}) + \sin\theta\,(\mathbf{x} \times \mathbf{n}) \\ &= \cos\theta\,\mathbf{x} + (1 - \cos\theta)\,(\mathbf{x} \cdot \mathbf{n})\,\mathbf{n} + \sin\theta\,(\mathbf{x} \times \mathbf{n}). \end{aligned}$$

f) If $\mathbf{n}$ and $\mathbf{x}$ are the *column* vectors $\mathbf{n} = (a,b,c)$, $\mathbf{x} = (x,y,z)$, and $\mathbf{n}^T$ is the transpose, show that as matrices

$$(\mathbf{x} \cdot \mathbf{n})\,\mathbf{n} = \mathbf{n}(\mathbf{n}^T\mathbf{x}) = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{x} \times \mathbf{n} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Thus, deduce our final formula:

$$R = (\cos\theta)I + (1 - \cos\theta) \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} + (\sin\theta) \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

g) Find the matrix that rotates $\mathbb{R}^3$ through an angle of θ using as axis the line through the origin and the point $(1,1,1)$.