Proof in Mathematics, Philosophy, and Law

Some Topics

Proof: Convince a dubious person. It depends on the audience.

1. I have a neck, a bottle has a neck, therefore I am a bottle.

2. **Proofs Without Understanding**

   Let $N := 15 \times 7 \times 8 \times 23 \times 41$.

   Is N divisible by 10?

   Solve $(x-3)(x+2)=0$ $[x^2+x-6=0]$.

   792120

3. **Impossible. Does not exist** How do you prove something is impossible? How do you prove something does not exist?

   $\sqrt{2}$ is irrational.

   Proof by contradiction. Law of the excluded middle.

   $\sin x$ is not a polynomial.

   Is there a real number x so that $x^2 = -1$?

   Is $\sqrt{i}$ a complex number? In other words, is there a complex number z so that $z^2 = i$?

   Leibniz (1646 – 1716) did not think so.

   [Solution: $z = \frac{1+i}{\sqrt{2}}$.]

4. **Prove Something Exists** How do you prove something exists if you can't exhibit it? Infinite number of primes.

   Euclid `http://primes.utm.edu/notes/proofs/infinite/`

   Here is a recent variant proof by Filip Saidak: "A New Proof of Euclid's Theorem," *Amer. Math. Monthly*, Vol. 113, No. 10, Dec 2006.

PROOF: Let $n > 1$ be a positive integer (such as $n = 2$). Since n and $n+1$ are consecutive integers, they are relatively prime [that is, they do not have a common prime factor]. Hence the number $N_2 := n(n+1)$ must have at least two different prime factors. Similarly, since the integers $n(n+1)$ and $n(n+1)+1$ are consecutive, and therefore relatively prime, the number $N_3 := n(n+1)[n(n+1)+1]$ must have at least 3 different prime factors. This can be continued indefinitely, so the number of primes is infinite.

Infinite number of primes of the form $4n+1$ and $4n-1$.

Bertrand: There is at least one prime between n and $2n$. For an "elementary" proof see

`https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate`

and

`http://www.math.uni-bonn.de/people/karcher/BertrandN_2N.pdf`

Does $x^5 - 2x - 13 = 0$ have a real root?

Note: $f(0) = -13,\ f(2) = 15$

Estimate root (crude; better: Newton)

Proof that establish existence but give no further information vs proof that yield algorithms for finding the object. Algorithms that run quickly are more useful.

EXAMPLE: Using the Brouwer Fixed Point Theorem

$$\begin{aligned} 2x + 5y &= \frac{x^2 + 7e^{\sin(2xy=1)}}{5 + 4x^2 - 3xy + 157y^2} - 9 \\ x + 3y &= 5 + 3\cos(x - 8y - 1) \end{aligned}$$

5. **Prove that two things are equal**

   $.999999... = 1$?

   Prove the PYTHAGOREAN THEOREM (using similar triangles).

GENERALIZATION: Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$

GENERALIZATION: Inner product in a vector space.

EULER'S FORMULA:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

For Euler's original proof see

George Polya, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*, Princeton University Press.

The number of integers equals the number of even integers.

6. **Who are the parents of proof?** Asking an interesting new question a high mark of creativity – even if you answer it INCORRECTLY .

   Using the concept of a sequence of real numbers $a_1, a_2, \ldots a_n, \ldots$ converging to the number A, Cauchy looked when a sequence of functions $f_k(x)$ converges to the function $f(x)$ at every point x of a finite interval, say $a \le x \le b$.

   EXAMPLE: The functions $f_k(x) = x^k$ converge to $f(x) \equiv 0$ at every point of the interval $0 \le x \le 1/2$.

   Cauchy "proved" that if the functions f_k are continuous, then the limit function $f(x)$ is also continuous.

   Then someone showed him the counterexample $f_k(x) = x^k$ on the interval $0 \le x \le 1$ converge to $f(x) \equiv 0$ at every point of the interval $0 \le x < 1$ while $f_k(1) = 1$ so this sequence of continuous functions converges to a *discontinuous* function.

   But it started everyone thinking of how to understand the phenomena, a real achievement.

7. **Secure communication. Secure signature.** You are in a remote country with email but not telephone contact. You want your

parents to deposit $5,000 in a local bank account there. How do you prove to them that the message is really from you and that it has not been tampered with?

The next few questions concern **Baysian Probability**

8. Say a 20 year old friend is tested for a relatively rare cancer that occurs in only 1 out of every 1,000 people her age. The test is 99% accurate in the sense that only 1% of those who do not have the cancer still test positive and 95% of those who have the cancer test positive.
   a) If your friend tests positive, what is the likelihood that she has the cancer?
   b) If your friend tests negative, what is the likelihood that she has the cancer?
   c) Repeat parts a) and b) if only 1 out of every 10,000 people her age have the cancer.

9. Your next-door neighbor has a rather old and temperamental burglar alarm. If someone breaks into his house, the probability of the alarm sounding is .95. In the last two years, though, the alarm has gone off on five different nights, each time for no apparent reason. Police records show that the chance of a home in your neighborhood being burglarized on any given night are 2 in 10,000. If your neighbor's alarm goes off tonight, what is the likelihood his house is being burglarized?

10. During a power blackout, 100 people are arrested on suspicion of looting. Each is given a polygraph test. From past experience it is known that the polygraph is 90% reliable when administered to a guilty suspect and 98% reliable when given to someone who is innocent. Suppose that of the 100 suspects, only 12 were actually

involved in any wrongdoing. What is the probability that a given suspect is innocent given that the polygraph says she is guilty?

11. [From THINKING FAST AND SLOW by D. Kahneman, Chapter 16]

    A cab was involved in a hit-and run accident at night.

    Two cab companies, the Green and the Blue, operate in the city.

    Data:

    - 85% of the cabs in the city are Green, 15% are Blue.
    - A witness identified the cab as Blue. The court tested the reliability of the witness under the circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.

    What is the probability that the cab that was involved in the accident was Blue rather than Green?

    There are two items of information: their *base rate* and the *imperfectly reliabile testimony of the witness*. In the absence of a witness, the probability of the guilty can being Blue is 15%, which is the base rate of the outcome.

12. **Mengele's Skull** Josef Rudolf Mengele was a German SS officer and a physician in the Nazi concentration camp Auschwitz. Prove that he was dead.

    `http://www.cabinetmagazine.org/issues/43/keenan_weizman.php`

    Forensic Evidence: Washington Post

    `http://www.washingtonpost.com/local/crime/justice-dept-fbi-to-review-u`
    `2012/07/10/gJQAT6DlbW_story.html`

13. **Proving You are Human** Brian Christian, the "Most Human Human" award.

http://www.theatlantic.com/magazine/archive/2011/03/mind-vs-machine/308386/5/

and his book *The Most Human Human*, New York, N.Y: Doubleday, 2011. ISBN 0-385-53306-3.

14. **Prove the Effectiveness of Medication** Special Lecture by Professor Susan Ellenberg(School of Medicine) Sept. 20

15. **Forensic Linguistics** by Jack Hitt, "Words on Trial" Can linguistics solve crimes that stump the police? *New Yorker*, July 23, 2012.

    http://proxy.library.upenn.edu:2081/ehost/detail?vid=5&hid=7&sid=119153d8-6dc2-4b13-84ba-5342a2876b28%40sessionmgr4&bdata=JnNpdGU9ZWhvc3QtbGl2ZQ%3d%3d#db=aph&AN=77872445

    A letter in response to Jack Hitt's article:

    The disagreement between experts in Jack Hitt2̆019s article about how to use linguistics to solve crimes illustrates the fact that, as is often the case in the other forensic disciplines, linguistics does not involve a set of common, repeatable, precise, and peer-reviewed methods that are grounded in theory and substantiated by a body of evidence ("Words on Trial," July 23rd).

    Even the more established forensic disciplines have no mandatory standards, certification, or accreditation. To make matters worse for the accused, forensic experts are most often trotted out in court to testify on behalf of the prosecution. This is because defendants frequently do not have the resources to commission an alternative forensic analysis, which puts them at a pronounced and unfair disadvantage. Linguistics experts, with a clearly unreliable set of methods, are subject to the same cognitive biases that color the judgment of criminal investigators, judges, and juries. This only weakens the process, by giving it a false air of objectivity, and further tilts the balance against the accused, who are supposedly deemed innocent until proven guilty. Until forensic linguistics emerges as a discipline rooted in scientific rigor, courts should reject it.

16. **A Mathematician Crunches the Supreme Court's Numbers**

    Patterns in Supreme Court Decisions by Lawrence Sirovich.

    `http://www.nytimes.com/2003/06/24/science/a-mathematician-crunches-the-`
    `html?pagewanted=all&src=pm`

    The second Rehnquist Court has remained unchanged in composition for 8 yr, resulting in a large temporally stable database. This paper reports on a mathematically objective analysis of this ensemble of rulings aimed at extracting key patterns and latent information. Although the rulings of a nine-justice Court require representation in nine dimensions, smaller spaces describe the Court's actions; e.g., a 2D subspace describes the margins of all decisions, and use of Shannon information shows that the Court acts as if composed of 4.68 ideal justices. Comparison is also made with the 1959–1961 and 1967-1969 Warren Courts. Both Warren Courts have remarkable parallels with the Rehnquist Court. In each instance, we present an optimal mapping of the justices between the Courts, which underscores the similarity in the workings of seemingly dissimilar courts.

    `http://www.pnas.org/content/100/13/7432.full.pdf`

17. **A Mathematical Challenge to Obesity**

    New York Times, May 14, 2012

    Carson Chow has used mathematical models to determine the causes of obesity, and ways to stem the epidemic.

    `http://www.nytimes.com/2012/05/15/science/a-mathematical-challenge-to-`
    `html`

18. **Do Cellphones Cause Brain Cancer?**

New York Times Apr 13, 2011 – Yes, no, maybe – the answer seems to change with every new study. Finding the definitive solution turns out to be a science in itself.

http://www.nytimes.com/2011/04/17/magazine/mag-17cellphones-t.html?pagewanted=all